The largest Kronecker and Littlewood–Richardson coefficients

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Joint work with Igor Pak and Greta Panova

Warmup: find two unrelated persons

Two unrelated persons (perhaps)

L. Kronecker

D. Littlewood

L. Kronecker

Kronecker & LR coefficients (the symmetric group)

$\lambda = (\lambda_1 \geq \cdots \geq \lambda_\ell \geq 1)$ partition of $n = \lambda_1 + \ldots + \lambda_\ell$

Irreducible representations of S_n: S^λ (Specht modules)

– **Kronecker** $g(\lambda, \mu, \nu)$ *tensor product decomposition multiplicity*:

$$S^\mu \otimes S^\nu = \bigoplus_{\lambda \vdash n} g(\lambda, \mu, \nu) S^\lambda \qquad (\lambda, \mu, \nu \vdash n)$$

– **LR** $c^\lambda_{\mu\nu}$ *induced multiplicity*: $\mathrm{Ind}^{S_n}_{S_k \times S_{n-k}} S^\mu \otimes S^\nu = \bigoplus_{\lambda \vdash n} c^\lambda_{\mu\nu} S^\lambda$

Via *Schur polynomials* $s_\lambda(x_1, \ldots, x_n) := \dfrac{\det[x_i^{\lambda_j + n - j}]}{\prod_{i<j}(x_i - x_j)}$

$$s_\lambda(x_i y_j) = \sum_{\mu,\nu} g(\lambda, \mu, \nu) s_\mu(x_i) s_\nu(y_j) \qquad s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda$$

Important in Algebraic Combinatorics: symmetric functions, Grassmannians.

Combinatorial interpretations are known for LR but not for Kronecker.

Stanley's problems

Theorem (Stanley (2015))

$$\max_{\lambda,\mu,\nu\vdash n} g(\lambda,\mu,\nu) = \sqrt{n!}\; e^{-O(\sqrt{n})}$$

$$\max_{\lambda\vdash n,\mu,\nu} c^{\lambda}_{\mu\nu} = 2^{n/2-O(\sqrt{n})}$$

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Problem (Stanley)

What partitions attain these maxima??????

Largest dimension

$f^\lambda = \dim S^\lambda$ (dimension of irreducible representation of S_n)

$= \#$ **standard Young tableaux** (SYT)

1	3	6
2	4	
5		

$= \dfrac{n!}{\prod_{\square\in\lambda} \text{hook}_\square}$ (hook-length formula) $f^{321} = \frac{6!}{1\cdot1\cdot1\cdot3\cdot3\cdot5} = 8$

(Old) Problem: Find the asymptotics of $D(n) := \max_{\lambda\vdash n} f^\lambda$

(Bivins et al (1954), Baer-Brock (1968), McKay (1978))

Largest dimension

Burnside-Frobenius identity:

$$\sum_{\lambda \vdash n} (f^\lambda)^2 = n! \implies \frac{\sqrt{n!}}{p(n)} \le \max_{\lambda \vdash n} f^\lambda \le \sqrt{n!}$$

$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{2n/3}}$ # partitions of n

(early wrong conjectures: $D(n) \ge \sqrt{n!}/n$ or $D(n) \ge \sqrt{n!}/poly(n)$)

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Theorem (Vershik-Kerov (1985))

$$\sqrt{n!}\ e^{-1.29\sqrt{n}} \le \max_{\lambda \vdash n} f^\lambda \le \sqrt{n!}\ e^{-0.11\sqrt{n}}$$

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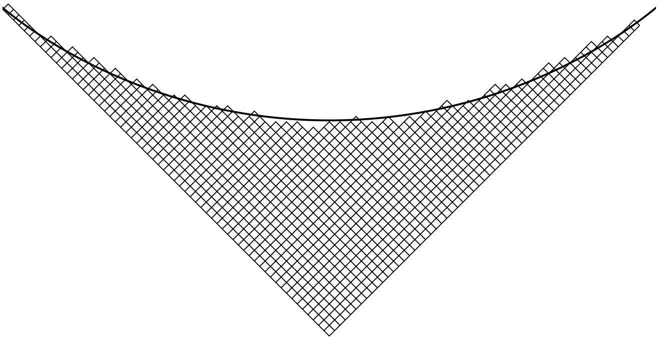
(Similar question) What partitions attain max dimension?

Partitions for largest (& typical) dimension

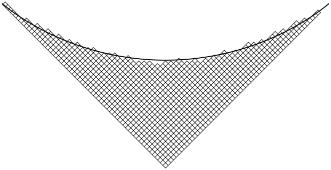
Vershik-Kerov–Logan-Shepp (VKLS) limit shape[1]



$$[\lambda] \to \omega(x) = \frac{2}{\pi}\left(x \arcsin(x/2) + \sqrt{4 - x^2}\right)$$

[1] pic from Romik's book; partition sampled from Plancherel measure $\frac{(f^\lambda)^2}{n!}$

Partitions attaining largest dimension



Partitions sequence $\lambda^{(n)} \vdash n$ is **Plancherel** if

$$f^{\lambda^{(n)}} \geq \sqrt{n!}\; e^{-c\sqrt{n}}$$

Theorem (Logan-Shepp (1977), Vershik-Kerov (1985))

Every Plancherel sequence has VKLS limit shape.

Gave solution to Ulam's problem on *longest increasing subsequences*, $\lambda_1 \sim 2\sqrt{n}$.

Max Kronecker are Plancherel

Theorem (Pak-Panova-Y.)

(i) $g(\lambda, \mu, \nu) = \sqrt{n!}\ e^{-O(\sqrt{n})} \implies \lambda, \mu, \nu$ *Plancherel*

(ii) $\forall$ *Plancherel* λ, μ $\exists$ *Plancherel* ν: $g(\lambda, \mu, \nu) = \sqrt{n!}\ e^{-O(\sqrt{n})}$

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Proof idea:

$$f^{\mu} f^{\nu} = \sum_{\lambda} g(\lambda, \mu, \nu) f^{\lambda}$$

$$\sum_{\lambda, \mu, \nu \vdash n} g(\lambda, \mu, \nu)^2 = \sum_{\alpha \vdash n} z_{\alpha} = n!(1 + O(1/n^2))$$

(z_{α} centralizer size.) Plus the bound: $g(\lambda, \mu, \nu) \leq f^{\lambda}$

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Vanishing: $\exists\ f^\lambda = n!^{\Theta(1)}, f^\mu, f^\nu = \sqrt{n!}e^{-O(\sqrt{n})}$ s.t. $g(\lambda,\mu,\nu) = 0$.
Follows from Dvir's theorem.

Conjecture. λ,μ,ν Plancherel $\implies g(\lambda,\mu,\nu) = \sqrt{n!}\ e^{-O(n)}$

Max LR coefficients

Theorem (Pak-Panova-Y)

$$\binom{n}{k}^{1/2} e^{-d\sqrt{n}} \leq \max_{\lambda \vdash n,\ \mu \vdash k,\ \nu \vdash n-k} c^{\lambda}_{\mu\nu} \leq \binom{n}{k}^{1/2}$$

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In fact,

$$\sum_{\lambda \vdash n} (c^{\lambda}_{\mu\nu})^2 \leq \binom{n}{k}, \qquad \sum_{\mu \vdash k, \nu \vdash n-k} (c^{\lambda}_{\mu\nu})^2 \leq \binom{n}{k}$$

$$\sum_{\lambda \vdash n, \mu \vdash k, \nu \vdash n-k} (c^{\lambda}_{\mu\nu})^2 \geq \binom{n}{k}$$

Max LR attained on Plancherel

Theorem (Pak-Panova-Y)

(i) $\forall$ *Plancherel* $\lambda \vdash n$ $\exists$ *Plancherel* $\mu \vdash k = n\theta$, $\nu \vdash n(1-\theta)$:

$$c^{\lambda}_{\mu\nu} = \binom{n}{k}^{1/2} e^{-O(\sqrt{n})}$$

(ii) Similarly, $\forall$ *Plancherel* μ, ν $\exists$ *Plancherel* λ *...*
(iii) $\forall$ *Plancherel* λ, μ $\exists$ ν *(with VKLS limit shape):*

$$f^{\nu} = \sqrt{n!}\ e^{-O(n^{2/3}\log n)} \qquad c^{\lambda}_{\mu\nu} = \binom{n}{k}^{1/2} e^{-O(n^{2/3}\log n)}$$

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Proof ideas: Estimates from the identities

$$\sum_{\lambda\vdash n} c^{\lambda}_{\mu\nu} f^{\lambda} = \binom{n}{k} f^{\mu} f^{\nu}, \qquad \sum_{\mu\vdash k,\nu\vdash n-k} c^{\lambda}_{\mu\nu} f^{\mu} f^{\nu} = f^{\lambda}$$

and skew SYT bounds + properties of VKLS shape for (iii).

Large LR implies large dimensions

Theorem (Pak-Panova-Y)

$\lambda \vdash n, \ \mu, \nu \vdash n/2$

$$c^{\lambda}_{\mu\nu} = \binom{n}{n/2}^{1/2} e^{-O(n/\log n)}$$

$$\implies f^{\lambda} = \sqrt{n!}\ e^{-O(n)}, \quad f^{\mu}, f^{\nu} = \sqrt{(n/2)!}\ e^{-O(n)}$$

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$$\implies f^{\lambda} = \sqrt{n!}\; e^{-O(n)}, \quad f^{\mu}, f^{\nu} = \sqrt{(n/2)!}\; e^{-O(n)}$$

Proof ideas:

$$\sum_{\lambda \vdash n} (c^{\lambda}_{\mu\nu})^2 = \sum_{\alpha,\beta,\gamma,\delta} c^{\mu}_{\alpha\gamma} c^{\mu}_{\alpha\delta} c^{\nu}_{\beta\gamma} c^{\nu}_{\beta\delta} \quad \text{(from skew Cauchy)}$$

$$c^{\lambda}_{\mu\nu} \leq e^{a\sqrt{n}} \max_{\alpha,\beta} c^{\mu}_{\alpha\beta} \max_{\alpha,\beta} c^{\nu}_{\alpha\beta}$$

$$f^{\lambda} \geq c^{\lambda}_{\mu\nu} f^{\mu} f^{\nu}, \qquad f^{\lambda} \geq e^{-un} (c^{\lambda}_{\mu\nu})^{\log_2 n}$$

Max LR with few rows

$$C(\ell, n) := \max_{\lambda \vdash n,\ \ell(\lambda)=\ell,\ \mu,\nu} c^{\lambda}_{\mu\nu}$$

Theorem (Pak-Panova-Y.)

$$n^{\ell^2/2 - a\ell} e^{-b\ell^2 \log \ell} \leq C(\ell, n) \leq (n+1)^{\ell^2/2}$$

Proof uses Knutson-Tao combinatorial interpretations.

Corollary

$$\log C(\ell_n, n) \sim \frac{1}{2}(\ell_n)^2 \log n, \qquad \ell_n = O(\sqrt{n}/\log n)$$

Vanishing LR

Theorem (Pak-Panova-Y)

$\mu, \nu \vdash n/2$ *Plancherel* $\exists\, \lambda$ *(with VKLS limit shape)*

$$f^\lambda = \sqrt{n!}\; e^{O(\sqrt{n}\log n)} \quad \& \quad c^\lambda_{\mu\nu} = 0.$$

Conjecture. $\exists$ Plancherel λ, μ, ν

$$\frac{1}{\sqrt{n}}\left(\frac{n}{2} - \log_2 c^\lambda_{\mu\nu}\right) \to \infty$$

Containment of max LR

Theorem (Lam-Postnikov-Pylyavskyy (2007))

$$c^{\lambda}_{\mu\nu} \leq c^{\lambda}_{\mu\cup\nu,\mu\cap\nu}$$

Corollary

$\forall\ \lambda\ \exists\ \mu \subseteq \nu \subset \lambda$

$$c^{\lambda}_{\mu\nu} = \max_{\alpha,\beta} c^{\lambda}_{\alpha,\beta}$$

Conjecture.

$$c^{\lambda}_{\mu\nu} = \max_{\alpha,\beta} c^{\lambda}_{\alpha,\beta} \implies \mu \supseteq \nu \text{ or } \mu \subseteq \nu$$

Monotinicity & stability of max LR

$$C(n,k) := \max_{\lambda\vdash n,\mu\vdash k,\nu\vdash n-k} c^{\lambda}_{\mu\nu} \quad C(n) := \max_k C(n,k) \quad D(n) = \max_{\lambda\vdash n} f^{\lambda}$$

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
$\mathbf{C}(n)$	1	1	1	1	1	2	2	2	2	3	3	4	4	5	6	8	9	11	12	18	24	32	35

12	1	1	1	2	3	3	4	3	3	2	1	1	1								
13	1	1	1	2	3	3	4	4	3	3	2	1	1	1							
14	1	1	1	2	3	3	4	5	4	3	3	3	1	1	1						
15	1	1	1	2	3	6	6	5	5	6	6	3	2	1	1	1					
16	1	1	1	2	3	6	8	7	6	7	8	6	3	2	1	1	1				
17	1	1	1	2	3	6	8	9	8	8	9	8	6	3	2	1	1	1			
18	1	1	1	2	3	6	8	11	10	9	10	11	8	6	3	2	1	1	1		
19	1	1	1	2	3	6	8	11	12	11	11	12	11	8	6	3	2	1	1	1	
20	1	1	1	2	3	6	8	11	12	13	18	13	12	11	8	6	3	2	1	1	1
21	1	1	1	2	3	6	16	12	14	14	24	24	14	14	12	16	6	3	2	1	1
22	1	1	1	2	3	6	16	20	15	16	27	32	27	16	15	20	16	6	3	2	1
23	1	1	1	2	3	6	16	20	24	19	30	35	35	30	19	24	20	16	6	3	2

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\mathbf{D}(n)$	1	1	2	3	6	16	35	90	216	768	2310	7700	21450	69498	292864	1153152

Monotinicity & stability of max LR

$$C(n,k) := \max_{\lambda \vdash n, \mu \vdash k, \nu \vdash n-k} c^{\lambda}_{\mu\nu} \quad C(n) := \max_k C(n,k) \quad D(n) = \max_{\lambda \vdash n} f^{\lambda}$$

Theorem (Pak-Panova-Y)

(i) $\{C(n)\}$ *is nondecreasing*

(ii) fix k*, then* $\{C(n,k)\}$ *is bounded and nondecreasing*

(iii) stability: for every k *and* $n \geq \binom{k+1}{2}$ *we have* $C(n,k) = D(k)$.

Conjecture. $C(n) < 2^{n/2} e^{-a\sqrt{n}}$

Skew SYT

Theorem (Pak-Panova-Y)

$$\sqrt{n!/m!}\; e^{-a\sqrt{n}} \le \max_{\lambda\vdash n,\ \mu\vdash m} f^{\lambda/\mu} \le \sqrt{n!/m!}\; e^{a\sqrt{n}}$$

Theorem (Pak-Panova-Y)

(i) $\lambda \vdash n$, $\mu \vdash m = n\theta$ *Plancherel, then:*

$$f^{\lambda/\mu} = \sqrt{n!/m!}\; e^{-O(n^{2/3}\log n)}$$

(ii) Suppose

$$f^{\lambda/\mu} = \sqrt{n!/(n/2)!}\; e^{O(n/\log n)}$$

then

$$f^{\lambda} = \sqrt{n!}\; e^{O(n)}, \qquad f^{\mu} = \sqrt{(n/2)!}\; e^{O(n)}$$

Generalizations to other groups

Most bounds/results extend to general finite groups with few conjugacy classes (our companion paper [PPY2])

R a k h m e t!

Thank you!