Ehrhart polynomial of some Schläfli simplices

Damir Yeliussizov

UCLA

Joint with Igor Pak

Southern California Discrete Math Symposium 2017

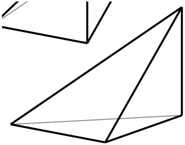
Schläfli simplex

$\mathbf{a} = (a_1, \ldots, a_n), \qquad \mathbf{S_a} \subset \mathbb{R}^n :$

$$1 \geq \frac{x_1}{a_1} \geq \cdots \geq \frac{x_n}{a_n} \geq 0$$

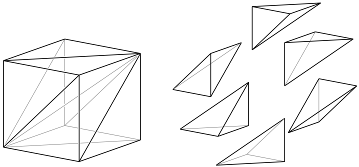
$$v : (0, \ldots, 0), (a_1, 0, \ldots, 0), (a_1, a_2, 0, \ldots, 0), \ldots, (a_1, a_2, \ldots, a_n)$$

$$\mathrm{Vol}(\mathbf{S_a}) = a_1 \cdots a_n / n!$$



Schläfli simplex

- L. Schäfli (18**) **orthoschemes** (Euclidean, Lobachevsky, spherical geometry). H. Coxeter (1991) named after Schläfli
- also known as **path simplex** (orthogonal edges form a path)
  Hadwiger's conjecture (1957): *Every simplex can be decomposed into a finite number of path-simplices*



Wikipedia Schläfli simplex

- also known as **lecture hall polytopes** ($\mathbf{a} \in \mathbb{N}^n$), Bousquet-Mélou & Eriksson (1997), Savage & Schuster (2012), ...

Schläfli simplex: some questions

Given $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$ (as an input in binary).

1. How many integer points in $\mathbf{S_a}$? $\#\left(\mathbf{S_a} \cap \mathbb{Z}^n\right)$
2. $t \in \mathbb{N}$, compute the **Ehrhart polynomial** $\mathcal{E}_{\mathbf{S_a}}(t) = \#\left(t\mathbf{S_a} \cap \mathbb{Z}^n\right)$

$P \subset \mathbb{R}^n$ lattice polytope, tP its t-dilation (vert. coord's $\times$ t)
Ehrhart polynomial: $\mathcal{E}_P(t) := \#\left(tP \cap \mathbb{Z}^n\right) = \mathrm{Vol}(P)/n!t^n + \ldots$

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- For a general n-dim simplex, computing $\mathcal{E}_P$ is **#P**-complete.
- For n fixed can be computed in polynomial time (Barvinok).
- Special easily computable case: If $a_1 = \cdots = a_n = 1$, then

$$\mathcal{E}_{\mathbf{S_a}}(t) = \#(t \geq x_1 \geq \cdots \geq x_n \geq 0) = \binom{t+n}{n}$$

- The problem is related to **integer partitions** (*Sylvester denumerant*):

$$p_{\mathbf{a}}(N) = \#\{(x_1, \ldots, x_n) : x_1 a_1 + \ldots + x_n a_n = N\}$$

Binary partitions

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recurrence: $q(2N+1) = q(2N), \quad q(2N+2) = q(2N+1) + q(N)$

$$\text{g.f.: } 1 + \sum_{N=1}^{\infty} q(N) t^N = \prod_{n \geq 0} \frac{1}{1 - t^{2^n}}.$$

Cayley's theorem

Theorem (Cayley, 1857)

The number of partitions of $2^x - 1$ *into the parts* $1, 1', 2, 2^2, \ldots, 2^{x-1}$ *is equal to the number of* x*-partitions (first part unity, no part greater than twice the preceding one).*

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$q(0) + \cdots + q(2^n - 1) =$
$\#\{(b_1, \ldots, b_n) \in \mathbb{N}^n : 1 \leq b_1 \leq 2, 1 \leq b_2 \leq 2b_1, \ldots, 1 \leq b_n \leq 2b_{n-1}\}$

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- Konvalinka & Pak (2014) found geometric bijective proof:

$$\varphi : (b_1, \ldots, b_n) \mapsto (2 - b_1, 2b_1 - b_2, \ldots, 2b_{n-1} - b_n)$$

Cayley & Schläfli

Cayley polytope $\mathbf{C}_n \subset \mathbb{R}^n$ (convex hull of Cayley compositions):

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Schläfli simplex $\mathbf{S}_n \subset \mathbb{R}^n$:

$$1 \geq \frac{x_1}{2} \geq \cdots \geq \frac{x_n}{2^n} \geq 0.$$

$\mathbf{C}_n \subset \mathbf{S}_n$, $\mathrm{Vol}(\mathbf{S}_n) = 2^{\binom{n+1}{2}}/n!$, where $2^{\binom{n+1}{2}}$ is the *total* number of labelled graphs on $n+1$ vertices.

Binary partitions: asymptotics

Mahler (1940), De Bruijn (1948), Knuth (1966):

$$\log q(N) \sim c \log^2 N$$

If the input N has bit size n, the output $q(N)$ has $O(n^2)$ bits

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$$\alpha_k = \Gamma\left(\frac{2\pi i k}{\log 2}\right) \zeta\left(1 + \frac{2\pi i k}{\log 2}\right) / \log 2$$

The results

$\mathbf{a} = (a_1, \ldots, a_n) = (c_1, c_1 c_2, \ldots, c_1 c_2 \cdots c_n) \in \mathbb{N}^n$ is a *factorial-type* sequence ($c_i \in \mathbb{N}$)

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Theorem (Pak, Y., 2017)

Given a factorial-type sequence $\mathbf{a} = (a_1, \ldots, a_n)$ and $s, t \in \mathbb{N}$ (all in binary). The following functions can be computed in polynomial time

- $\mathcal{E}_{\mathbf{S_a}}(t) = \# \left(t\mathbf{S_a} \cap \mathbb{Z}^n \right)$
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Corollary

Special case: For $\mathbf{a} = (1, 2, 4, \dots, 2^{n-1})$, the functions $\mathcal{E}_{\mathbf{S_a}}$ can be computed in polynomial time

Application to integer partitions

Given $\mathbf{a}$ and N, count integer partitions (also known as Sylvester denumerant problem)

$$p_{\mathbf{a}}(N) := \#\{(x_1, \ldots, x_n) \in \mathbb{N}^n : a_1 x_1 + \cdots + a_n x_n = N\}$$

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Proof idea: *affine transformations* between corresponding lattice polytopes living inside Schläfli and partition simplices. The problem is then reduced to computing the function $\mathcal{E}_{\mathbf{S}_{\mathbf{a}}}(s, t)$.

Algorithm for computing $q(N)$, the number of binary partitions

$\beta : \mathbb{Z} \to \mathbb{Z}$ is given by $s \mapsto \lceil s/2 \rceil$ $\quad \Delta_{n,t} f(t) := \sum_{\ell=1}^{n} (-1)^{\ell-1} \binom{n}{\ell} f(t-\ell)$

$$e_n(t) = \sum_{\ell=1}^{2t} e_{n-1}(\ell), \qquad e_1(t) = \begin{cases} 1, & \text{if } s \leq 2t; \\ 0, & \text{otherwise.} \end{cases}$$

$$e_2(t) = \sum_{\ell=1}^{2t} e_1(\ell) = \sum_{\ell=\lceil s/2 \rceil}^{2k} 1 = \max(2t - \beta(s) + 1, 0)$$

Input: N in binary
Set $s = 2^n - N$ for n so that $2^n - N > 0$.
For $\ell = 1, \ldots, n$
    For $t = \beta^{\ell}(s), \ldots, \beta^{\ell}(s) + \ell - 1$
        $e_\ell(t) = e_\ell(t-1) + e_{\ell-1}(2t-1) + e_{\ell-1}(2t)$
    For $t = \beta^{\ell}(s) + \ell, \ldots, \beta^{\ell}(s) + 2\ell$
        $e_\ell(t) = \Delta_{\ell,t} e_\ell(t)$
Output: $q(N) = e_n(1)$.

Open problem

Problem. Given N (in binary). Compute the number of partitions of N into Fibonacci numbers in $poly(\log N)$ time.

Note: For partitions into *distinct parts* this problem was solved, Robbins (1996), Englund (2001).

Rahmet!