Schur operators and skew stable Grothendieck polynomials

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Headliner

Grothendieck polynomials skewed. There will be ...

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Skew Cauchy (main hero)

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Skew Pieris

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Grothendieck polynomials skewed. There will be ...

Skew Cauchy (main hero)

Skew Pieris

Catalan numbers?
Dual filtered graphs
Random walk on Young's lattice

Enumerative formulas but you won't see them

Schur operators on Young diagrams λ

$$u_i \cdot \lambda = \begin{cases} \lambda + \square \text{ on } i\text{-th column}, & \text{if possible,} \\ 0, & \text{otherwise} \end{cases}$$

$$d_i \cdot \lambda = \begin{cases} \lambda - \square \text{ on } i\text{-th column}, & \text{if possible,} \\ 0, & \text{otherwise} \end{cases}$$

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Ex: 

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Ex: $u_2 \cdot$  $=$  $\quad d_2 \cdot$  $=$ 

non-local commutativity ($[a, b] = ab - ba$ commutator)

$$[u_j, u_i] = 0, \quad [d_j, d_i] = 0, \quad |i - j| \geq 2$$

local Knuth relations

$$[u_{i+1}u_i, u_i] = 0, \quad [u_{i+1}u_i, u_{i+1}] = 0; \quad [d_i d_{i+1}, d_i] = 0, \quad [d_i d_{i+1}, d_{i+1}] = 0.$$

conjugate relations

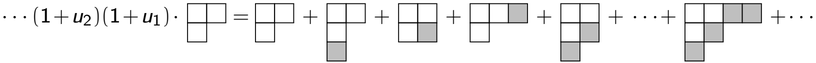
$$[d_j, u_i] = 0 \ (i \neq j), \quad d_{i+1}u_{i+1} = u_i d_i, \quad d_1 u_1 = 1.$$

Schur functions from Schur operators

$$A(x) := \cdots (1 + xu_2)(1 + xu_1) \quad B(x) := (1 + xd_1)(1 + xd_2) \cdots$$

$$\cdots A(x_2)A(x_1) \cdot \mu = \sum_{\lambda} s_{\lambda/\mu}(x_1, x_2, \ldots) \cdot \lambda,$$

$$\cdots B(x_2)B(x_1) \cdot \lambda = \sum_{\mu} s_{\lambda/\mu}(x_1, x_2, \ldots) \cdot \mu$$



$$A(x) \cdot (\mu) = \sum x^{|\text{hor. strip}|} \cdot (\mu + \text{hor. strip}), \quad B(x) \cdot (\lambda) = \sum x^{|\text{hor. strip}|} \cdot (\lambda - \text{hor. strip})$$

Non-local and local commutation relations for (u_i) imply

$$[A(x_i), A(x_j)] = 0$$

Hence $s_{\lambda/\mu}$ is (automatically) a **symmetric** function

Grothendieck-Schur operators [Y. 2017]

$$\widetilde{u}_i = u_i(1-\beta d_i) = u_i - \beta u_i d_i \qquad \text{(β loops added)}$$

$$\widetilde{d}_i = (1-\beta d_i)^{-1} d_i = d_i + \beta d_i^2 + \beta^2 d_i^3 + \cdots \qquad \text{(remove while can)}$$

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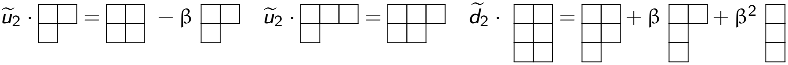

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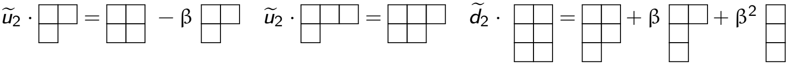

$\beta = 0$ gives Schur operators u, d

non-local commutativity: $[\widetilde{u}_i, \widetilde{u}_j] = [\widetilde{d}_i, \widetilde{d}_j] = 0, \quad |i - j| \geq 2$

local commutativity: $[\widetilde{u}_{i+1}\widetilde{u}_i, \widetilde{u}_i + \widetilde{u}_{i+1}] = 0, \quad [\widetilde{d}_i\widetilde{d}_{i+1}, \widetilde{d}_i + \widetilde{d}_{i+1}] = 0$

conjugate relations: $[\widetilde{u}_i, \widetilde{d}_j] = 0, \quad |i - j| \geq 2, \quad [\widetilde{u}_{i+1}, \widetilde{d}_i] = 0, \quad \widetilde{d}_1\widetilde{u}_1 = 1.$

Generally, relations are more complicated than for Schur operators

Symmetric skew Grothendieck polynomials

$$A(x) = \cdots (1 + x\widetilde{u}_2)(1 + x\widetilde{u}_1) \quad B(x) = (1 + x\widetilde{d}_1)(1 + x\widetilde{d}_2)\cdots$$

$$\cdots A(x_2)A(x_1)\cdot\mu = \sum_{\lambda} G^{\beta}_{\lambda /\!\!/ \mu}(x_1, x_2\ldots)\cdot\lambda$$

$$\cdots B(x_2)B(x_1)\cdot\lambda = \sum_{\mu} g^{\beta}_{\lambda/\mu}(x_1, x_2, \ldots)\cdot\mu$$

$$A(x)\cdot(\mu) = \sum_{\lambda/\mu \text{ hor. strip}} (1-\beta x)^{\mathrm{smth}} x^{|\lambda/\mu|}\cdot(\lambda), \quad B(x)\cdot(\lambda) = \sum_{\mu\subset\lambda} \beta^{\mathrm{smth}} x^{\#\mathrm{col's}(\lambda/\mu)}\cdot(\mu)$$

Commutation relations imply

$$[A(x_i), A(x_j)] = [B(x_i), B(x_j)] = 0$$

Hence $G^{\beta}_{\lambda /\!\!/ \mu}, g^{\beta}_{\lambda/\mu}$ are **symmetric** functions

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Why $G^{\beta}_{\lambda /\!\!/ \mu}$ and not $G^{\beta}_{\lambda/\mu}$? Because $G^{\beta}_{\lambda /\!\!/ \lambda} \neq G^{\beta}_{\varnothing} = 1$

Symmetric Grothendieck polynomials

$\{\widetilde{u}\}$ give Buch's formula (2002), $\mu = \varnothing$

$$G_\lambda^\beta = \sum_{T \in SVT(\lambda)} (-\beta)^{|T|-|\lambda|} x_1^{\#1\text{'s in } T} x_2^{\#2\text{'s in } T} \cdots$$

Set-valued tableaux (SVT):

$$T = \begin{array}{|c|c|c|} \hline 12 & 235 & 6 \\ \hline 35 & 68 & \\ \cline{1-2} \end{array} \qquad G_{(32)} = \cdots + (-\beta)^{10-5} x_1 x_2^2 x_3^2 x_5^2 x_6^2 x_8 + \cdots$$

G_λ first studied by Fomin-Kirillov (1996)

$$G_\lambda \in \hat{\Lambda} \qquad G_\square = e_1 - \beta e_2 + \beta^2 e_3 - \cdots \qquad G_\lambda = s_\lambda + \text{higher degree terms}$$

Buch (2002): G_λ analog of s_λ in K-theory of Grassmannian $\mathrm{Gr}_k(\mathbb{C}^n)$ of k-planes in $\mathbb{C}^n$. Product has **finite** decomposition + LR rule:

$$G_\lambda G_\mu = \sum_\nu c_{\lambda\mu}^\nu G_\nu, \qquad \Gamma = \bigoplus_\lambda \mathbb{Z} \cdot G_\lambda$$

Dual Grothendieck basis

$\widetilde{d}$ give dual formula of Lam-Pylyavskyy (2007)

$$g_\lambda = \sum_{T \in RPP(\lambda)} \beta^{|\lambda|-c(\lambda)} x_1^{\#\text{col's of } T \text{ with } 1} x_2^{\#\text{col's of } T \text{ with } 2} \cdots$$

Reverse plane partitions (RPP):

$$T = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 3 \\ \hline 1 & 2 & 2 \\ \cline{1-3} 1 & 2 \\ \cline{1-2}\end{array} \qquad g_{432} = \cdots + \beta^{9-5} x_1^2 x_2^2 x_3 + \cdots$$

$\{g_\lambda\}$ inhomogeneous basis of Λ and $g_\lambda = s_\lambda +$ lower degree terms

$$\langle g_\lambda, G_\mu \rangle = \delta_{\lambda\mu} \iff \sum_\lambda g_\lambda(x) G_\lambda(y) = \prod_{i,j} (1 - x_i y_j)^{-1} \text{ Cauchy identity}$$

Skew Cauchy identity

Theorem [Y. 2017]

$$\sum_{\lambda} G^{\beta}_{\lambda /\!\!/ \mu}(\mathbf{x}) g^{\beta}_{\lambda/\nu}(\mathbf{y}) = \prod_{i,j} (1 - x_i y_j)^{-1} \sum_{\kappa} G^{\beta}_{\nu /\!\!/ \kappa}(\mathbf{x}) g^{\beta}_{\mu/\kappa}(\mathbf{y})$$

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$$\Longleftrightarrow \; B(y)A(x) = (1 - xy)^{-1} A(x)B(y) \qquad (\textit{comm.relation})$$

Skew Cauchy identity

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$$\sum_{\lambda} G^{\beta}_{\lambda/\!/\mu}(\mathbf{x}) g^{\beta}_{\lambda/\nu}(\mathbf{y}) = \prod_{i,j} (1 - x_i y_j)^{-1} \sum_{\kappa} G^{\beta}_{\nu/\!/\kappa}(\mathbf{x}) g^{\beta}_{\mu/\kappa}(\mathbf{y})$$

$$\iff B(y)A(x) = (1 - xy)^{-1} A(x) B(y) \qquad (comm.relation)$$

$\beta = 0$ gives Schur case, Fomin's approach (1996).

For Schur was first given by Zelevinsky in Russian translation of Macdonald's book. Then generalized for many: Schur PQ, Hall-Littlewood, Macdonald, ...

Skew Cauchy identities are important for stochastic models, Schur and Macdonald processes (works of Borodin, Okounkov). Borodin-Petrov (2016): it became central and is directly related to Yang-Baxter equation.

Skew Cauchy encapsulates many properties of corresponding symmetric functions and can be viewed as a generalized RSK.

Applications of skew Cauchy

Skew Pieri rules

Various identities

Dual graphs

Probabilistic models

Enumerative formulas

Skew Pieri rules

Skew Cauchy (+ some work) gives skew Pieri

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Skew Cauchy (+ some work) gives skew Pieri

Theorem (Y. 2017)

$$g_{(k)} g_{\mu/\nu} = \sum_{\substack{\lambda/\mu \text{ hor strip} \\ \nu/\eta \text{ vert strip}}} w^{\lambda,\mu}_{\nu,\eta}(\beta)\ g_{\lambda/\eta}, \quad w = \pm\beta^{|\lambda/\mu|+|\nu/\eta|-k}\binom{a(\mu,\lambda)-b(\eta,\nu)-\nu/\eta}{|\lambda/\mu|+|\nu/\eta|-k}$$

$$G_{(1^k)} G_{\mu/\!\!/\nu} = \sum_{\substack{\lambda/\mu \text{ vert strip} \\ \eta\subset\nu}} W^{\lambda,\mu}_{\nu,\eta}(\beta)\ G_{\lambda/\!\!/\eta} \qquad W \text{ smth else but not hard}$$

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Skew Cauchy (+ some work) gives skew Pieri

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Corollary. Simple skew Pieri ($\beta = 1$):

$$g_{\square} g_{\mu/\nu} = (-i(\mu) + i(\nu)) g_{\mu/\nu} + \sum_{\lambda=\mu+\square} g_{\lambda/\nu} - \sum_{\eta=\nu-\square} g_{\lambda/\eta}$$

$$G_{\square} G_{\mu/\!\!/\nu} = \sum_{\substack{\lambda/\mu \text{ rook strip} \\ \eta\subset\nu}} (-1)^{|\lambda/\mu|} G_{\lambda/\!\!/\eta}$$

Corollary. $\beta = 0$ gives Assaf-McNamara's (2011) skew Pieri rule for Schur functions

$$s_{(k)} s_{\mu/\nu} = \sum_{\substack{\lambda/\mu \text{ hor strip} \\ \nu/\eta \text{ vert strip}}} (-1)^{|\nu/\eta|} s_{\lambda/\eta}$$

Applications: Some identities

$$\sum_{\lambda} q^{c(\lambda/\nu)} \beta^{|\lambda/\nu| - c(\lambda/\nu)} G_\lambda^\beta = \prod_i \frac{1}{1 - q x_i} G_\nu^\beta$$

$$\sum_{\lambda/\mu \text{ hor. strip}} (1 - \beta q)^{a(\mu,\lambda)} q^{|\lambda/\mu|} g_\lambda^\beta = \prod_i \frac{1}{1 - q y_i} g_\mu^\beta$$

$$\sum_{\lambda} G_{\lambda /\!\!/ \mu} = \prod_i \frac{1}{1 - x_i} \qquad (\beta = 1)$$

Recall in contrast classical Schur case:

$$\sum_{\lambda} s_{\lambda/\mu} = \prod_i \frac{1}{1 - x_i} \prod_{i<j} \frac{1}{1 - x_i x_j} \sum_{\kappa} s_{\mu/\kappa}$$

Applications: a curious identity. Catalan?

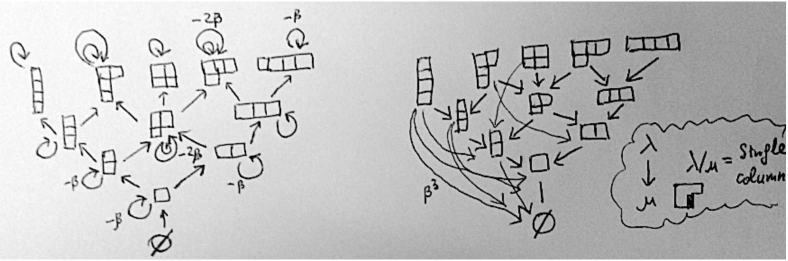
$$\sum_{\lambda} G_{\lambda/\delta_n} = \mathrm{Cat}_n \prod_i \frac{1}{1-x_i}, \quad \delta_n = (n, n-1, \ldots, 1)$$

Note: G_{λ/δ_n} not $G_{\lambda/\!\!/\delta_n}$

Applications: dual filtered graphs

$$up : \widetilde{U} = \widetilde{u}_1 + \widetilde{u}_2 + \cdots \qquad down : \widetilde{D} = \widetilde{d}_1 + \widetilde{d}_2 + \cdots$$

Skew Cauchy gives: $\quad [\widetilde{D}, \widetilde{U}] = \widetilde{D}\widetilde{U} - \widetilde{U}\widetilde{D} = 1$



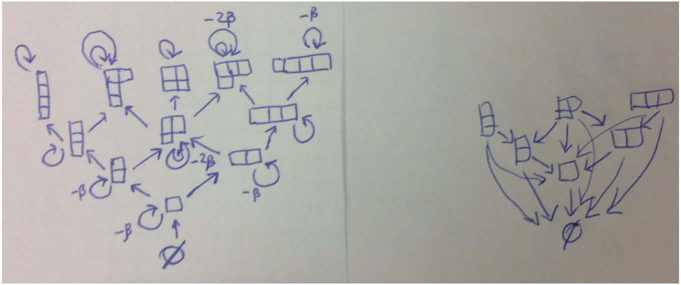
Stanley (1988) studied differential posets and Fomin (1997) dual graded graphs

Patrias-Pylyavskyy (2015) introduced dual filtered graphs as a K-theoretic analog

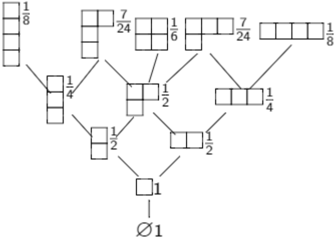
Applications: dual filtered graphs

$$up : \widetilde{U} = \widetilde{u}_1 + \widetilde{u}_2 + \cdots \qquad down : \overline{D} = -1 + (1 + \widetilde{d}_1)(1 + \widetilde{d}_2)\cdots$$

Skew Cauchy gives: $\quad [\overline{D}, \widetilde{U}] = \overline{D}\,\widetilde{U} - \widetilde{U}\,\overline{D} = 1 + \overline{D}$



Random walk on Young lattice



Vershik-Kerov: ϕ is **harmonic**

$$\phi(\mu) = \sum_{\lambda=\mu+\Box} \varkappa(\mu,\lambda)\phi(\lambda) \quad \varkappa(\mu,\lambda) \geq 0$$

$\dim_p(\lambda/\mu) =$ Prob $\mu \to \lambda$ after $|\lambda/\mu|$ steps.

$$\dim_p(\lambda/\mu) = \sum_{\mu=\nu_0\to\nu_1\ldots\to\nu_{|\lambda/\mu|}=\lambda} \prod_{i=0}^{|\lambda/\mu|-1} \frac{1}{i(\nu_i)+1}, \qquad i(\lambda) = \#\text{ inner corners } \lambda.$$

Theorem (Y. 2017) (i) $\dim_p(\lambda/\mu)$ is harmonic with $\varkappa(\mu,\lambda) = \frac{1}{1+i(\mu)}$. For $|\lambda| = n$, $\dim_p(\lambda)$ is a probability distribution.
(ii) Let $\rho : \Lambda \to \mathbb{C}$ be a homomorphism (specialization) with $\rho(g_\Box) = 1$. Then $\varphi(\lambda) := \rho(g_\lambda)$ is a harmonic function for same $\varkappa$.

That's it but there is more ...