ON EQUIDISTRIBUTION THEOREM FOR PLANE PARTITIONS

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Abstract. We prove equidistribution of two pairs of statistics on boxed plane partitions: (volume, trace) and (corner-hook volume, number of corners). The proof relies on different 3d visualizations of the corresponding non-intersecting path systems. In particular, we obtain a new visual proof for a volume generating function of plane partitions. We also introduce a new statistic called the cohook area on ordinary partitions, and prove that it is equidistributed with the area of partitions.

1. Introduction

Equidistributed statistics in combinatorics can be quite nontrivial and interesting. A remarkable example is MacMahon's theorem on equidistribution of the number of inversions and the major statistic on permutations [Mac16a], which has a conceptual bijective proof due to Foata [Foa68], see also [Sta11, Prop. 1.4.6].

In this paper, we study an equidistribution result of a similar kind but for statistics on plane partitions, which is a rather unusual instance compared to permutations.

1.1. Plane partitions.

A *plane partition* is a matrix $\pi = (\pi_{i,j})_{i,j\geq 1}$ of nonnegative integers with finitely many nonzero entries such that

$$\pi_{i,j} \geq \pi_{i+1,j}, \qquad \pi_{i,j} \geq \pi_{i,j+1}, \qquad \text{for all } i,j \geq 1.$$

Plane partition can be identified with its diagram

$$D(\pi) := \{(i,j,k) : 1 \leq k \leq \pi_{i,j}\},$$

which can be visually represented as a pile of 3d boxes, see Figure 1. We define the set of *corners* of π as follows:

$$\mathrm{Cor}(\pi) := \{(i,j,k) \in D(\pi) : (i+1,j,k), (i,j+1,k) \notin D(\pi)\}.$$

There are two natural statistics on plane partitions:

$$\text{the } \textit{volume } |\pi| := \sum_{i,j} \pi_{i,j} = |D(\pi)| \text{ and the } \textit{trace } \mathrm{tr}(\pi) := \sum_i \pi_{i,i}.$$

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There are also two other statistics on plane partitions introduced in [Yel21a, Yel21b]:

$$\text{the } \textit{corner-hook volume } |\pi|_{ch} := \sum_{(i,j,k)\in \mathrm{Cor}(\pi)} (i+j-1) \text{ and } \mathrm{cor}(\pi) := |\mathrm{Cor}(\pi)|.$$

The set of plane partitions whose diagram lies inside the box $[a] \times [b] \times [c]$ is denoted by $\mathrm{PP}(a, b, c)$, where we use the notation $[n] := \{1, \dots, n\}$.

We prove that the bivariate statistics (*volume*, *trace*) and (*corner-hook volume*, *number of corners*) are jointly equidistributed over boxed plane partitions.

Theorem 1.1 (Equidistribution of (*volume*, *trace*) and (*corner-hook volume*, *corners*))**.** *We have:*

$$\sum_{\pi \in \mathrm{PP}(a,b,c)} q^{|\pi|} t^{\mathrm{tr}(\pi)} = \sum_{\pi \in \mathrm{PP}(a,b,c)} q^{|\pi|_{ch}} t^{\mathrm{cor}(\pi)}.$$

Equivalently, for all n, k we have:

$$|\{\pi \in \mathrm{PP}(a, b, c) : |\pi| = n, \mathrm{tr}(\pi) = k\}| = |\{\pi \in \mathrm{PP}(a, b, c) : |\pi|_{ch} = n, \mathrm{cor}(\pi) = k\}|.$$

A weaker (as an infinite sum when $c \to \infty$) version of this theorem was established in [Yel21b] by an indirect argument via maps to matrices. In this paper, we show that a stronger result holds for a finite boxed version, by a direct combinatorial argument.

The following volume-trace generating function for plane partitions was obtained by Stanley [Sta73]:

$$\sum_{\pi \in \mathrm{PP}(a,b,\infty)} q^{|\pi|} t^{\mathrm{tr}(\pi)} = \prod_{i=1}^{a} \prod_{j=1}^{b} (1 - t q^{i+j-1})^{-1},$$

which generalizes the volume generating function (for $t = 1$) by MacMahon [Mac16b]. Note also that for $t = 1$ and a finite boxed version there is a nice product formula for the corresponding generating function

$$\sum_{\pi \in \mathrm{PP}(a,b,c)} q^{|\pi|} = \prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{1 - q^{i+j+k-1}}{1 - q^{i+j+k-2}}.$$

In the same series of works, MacMahon also conjectured volume generating functions for d-dimensional partitions which naturally generalize plane partitions in higher dimensions. Later, in [ABMM67] his conjecture was shown to be incorrect for all $d \geq 3$. In [Yel21a, Yel21b] the second author introduced the corner-hook volume on plane partitions (named 'up-hook volume' there up to a diagram rotation) and showed that it gives product formulas as in the volume generating function. We generalized this result to d-dimenisonal partitions in [AY23], and it turns out that the corner-hook volume is the correct statistic lying behind MacMahon's generating functions for general d. Theorem 1.1 shows that the two statistics are equidistributed for $d = 2$, which does not happen in higher dimensions.

In fact, we show a direct combinatorial proof of a more general identity with weights relating two known families of symmetric functions (indexed by ordinary partitions):

Schur polynomials s_λ and *dual stable Grothedieck polynomials* g_λ which can be viewed as K-theoretic extensions of Schur functions, see § 2.3, 2.4 for definitions and context. It is obtained by using the Lindström–Gessel–Viennot lemma [Lin73, GV89] and interpreting plane partitions as non-intersecting path systems in two different ways on the same picture, as in Figure 2. We then apply it to double enumeration of plane partitions.

The proofs use ideas developed by us in [AY22].

1.2. **Ordinary partitions.** As a byproduct of our approach, we also introduce a new statistic $|\cdot|_c$ for ordinary partitions, which we call the *cohook area* and show that it is equidistributed with the usual area $|\cdot|$ of partitions.

Theorem 1.2. *We have:*

$$\sum_{\lambda \in \mathrm{P}(a,b)} q^{|\lambda|} t^{d(\lambda)} = \sum_{\lambda \in \mathrm{P}(a,b)} q^{|\lambda|_c} t^{\mathrm{cor}(\lambda)},$$

where sum runs over all partitions λ, $d(\lambda)$ is the length of the Durfee square of λ, and $\mathrm{cor}(\lambda)$ *is the number of corners of λ.*

2. Preliminaries

2.1. **Partitions.** A *partition* is a sequence $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ of positive integers $\lambda_1 \geq \ldots \geq \lambda_\ell$, where $\ell(\lambda) = \ell$ is the *length* of λ. Denote $|\lambda| = \sum_i \lambda_i$ the *size* or *area* of λ. Every partition λ can be represented as the *Young diagram* $D(\lambda) := \{(i,j) : i \in [1,\ell], j \in [1,\lambda_i], (i,j) \in \mathbb{N}^2\}$. By λ' we denote the *conjugate partition* of λ, i.e. transpose of its diagram. For a cell $(i,j) \in D(\lambda)$ the values $\lambda_i - i + 1$ and $\lambda'_j - j + 1$ are called the *arm* and *leg* lengths of (i,j). Together, they constitute (i,j) hook length $h_\lambda(i,j) := (\lambda_i - i) + (\lambda'_j - j) + 1$.

A *Durfee square side* $d(\lambda)$ of partition λ is the largest k for which $\lambda_k \geq k$. The Frobenius notation $(a_1, \ldots, a_k | b_1 \ldots, b_k)$ of a partition λ with Durfee square side equal $k := d(\lambda)$ expresses the partition in terms of its hooks for $(i,i) \in D(\lambda)$, namely the arm $a_i = |\{(i,i), \ldots, (i,\lambda_i)\}|$ and the leg $b_i = |\{(i,i), \ldots, (\lambda'_i, i)\}|$. Clearly, $a_1 > \ldots > a_k \geq 1$ and $b_1 > \ldots > b_k \geq 1$ and $\sum_{i=1}^k a_i + b_i - 1 = |\lambda|$ is the partition area.

The set of partitions whose Young diagram lies inside the box $[a] \times [b]$ is denoted by $\mathrm{P}(a,b)$.

2.2. **Plane partitions.** A *plane partition* is a matrix $\pi = (\pi_{i,j})_{i,j \geq 1}$ of nonnegative integers with finitely many nonzero entries such that

$$\pi_{i,j} \geq \pi_{i+1,j}, \qquad \pi_{i,j} \geq \pi_{i,j+1}, \qquad \text{for all } i,j \geq 1.$$

Every plane partition π can be represented as the diagram

$$D(\pi) := \{(i,j,k) : 1 \leq k \leq \pi_{i,j}\},$$

which can be visually represented as a pile of 3d boxes, see Figure 1.

The *shape* of π is given by $\mathrm{sh}(\pi) = \{(i,j) : \pi_{i,j} > 0\}$. By the *side shape* $\mathrm{sh}_1(\pi)$ we denote the partition given by its first row $(\pi_{1,i})$.

An element $(i,j,k) \in D(\pi)$ is called *a corner*, if

$$(i+1,j,k) \notin D(\pi) \quad \text{and} \quad (i,j+1,k) \notin D(\pi).$$

If additionally $(i,j,k+1) \notin D(\pi)$ we call it *top corner*. The set of all corners is denoted by $\mathrm{Cor}(\pi)$ and its number $\mathrm{cor}(\pi) = |\mathrm{Cor}(\pi)|$. Alternatively,

$$\mathrm{Cor}(\pi) = \{(i,j,k) : \max(\pi_{i+1,j}, \pi_{i,j+1}) < k \leq \pi_{i,j}\}.$$

The *volume* $|\pi|$ and the *corner-hook volume* $|\pi|_{ch}$ of π are defined as follows:

$$|\pi| = \sum_{i,j\geq 1} \pi_{i,j}, \qquad |\pi|_{ch} = \sum_{(i,j,k)\in \mathrm{Cor}(\pi)} (i+j-1).$$

The *trace* of π is $\mathrm{tr}(\pi) := \sum_{i\geq 1} \pi_{i,i}$, i.e. the sum of diagonal entries.

The set of plane partitions whose diagram lies inside the box $[a] \times [b] \times [c]$ is denoted by $\mathrm{PP}(a,b,c)$ We regard plane partition $\pi \in \mathrm{PP}(a,b,c)$ as a pile of boxes.

For example, consider the plane partition π of shape $\mathrm{sh}(\pi) = (3,3,2,1)$ and the side shape $\mathrm{sh}_1(\pi) = (3,3,1)$ given by

$$\pi = \begin{pmatrix} 3 & 3 & 1 \\ 3 & 2 & 1 \\ 3 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix} \in \mathrm{PP}(4,3,3), \tag{1}$$

which is displayed in Figure 1. It has $\mathrm{tr}(\pi) = 5$, $|\pi| = 21$ and $|\pi|_{ch} = 3+5+4+5+5+5 = 27$, where

$$\mathrm{Cor}(\pi) = \{(1,2,3), (2,3,1), (3,1,3), (3,2,2), (3,2,1), (4,1,1)\}, \qquad \mathrm{cor}(\pi) = 6.$$

Remark 1. Definitions of corners follows conventions in [AY23]. It is equivalent to the notion of 'descents' in [Yel21a, Yel21b].

2.3. **Schur polynomials.** We denote $\mathbf{x}_n = (x_1, \ldots, x_n)$ (and similarly for other sets of variables). The Schur polynomials $\{s_\lambda\}$ can be defined as follows:

$$s_\lambda(\mathbf{x}_n) := \sum_{\pi \in \mathrm{SPP}(\lambda)} \prod_{(i,j)\in D(\lambda)} x_{\pi_{i,j}}$$

where $\mathrm{SPP}(\lambda)$ is the set of *column-strict plane partitions* of shape λ, i.e. filling of a diagram $D(\pi)$ with entries $1, \ldots, n$ so that the rows are weakly decreasing and columns are strictly decreasing. They satisfy the following determinantal formula.

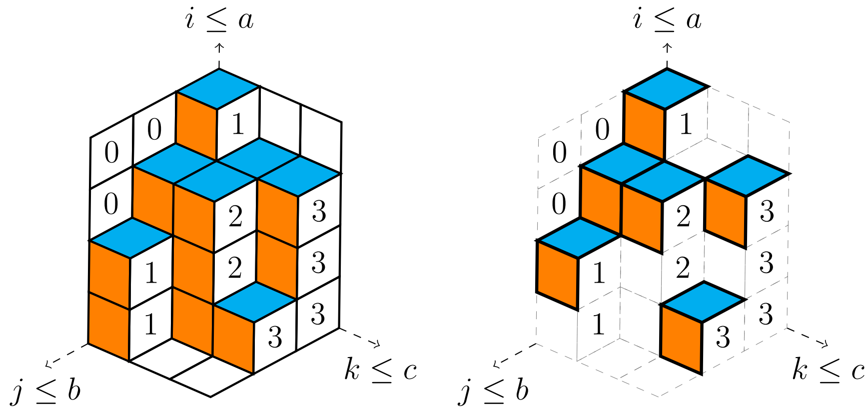


FIGURE 1. The plane partition π from (1) represented as a pile of boxes. We present it in a rotated way, so that the side shape $\mathrm{sh}_1(\pi) = (3,3,1)$ appears at the bottom with rows corresponding to y-direction, columns to z-direction and height to x-direction (compared to Fig. 2). The numbers on the right sides of boxes represent the heights $\pi_{i,j}$.

Proposition 2.1 (Nägelsbach–Kostka or dual Jacobi–Trudi identity)**.** *For $n \ge 1$ and $\ell(\lambda) \le n$ we have:*

$$s_\lambda(\mathbf{x}_n) = \det\left(e_{\lambda'_i - i + j}(\mathbf{x}_n)\right)_{i,j\in[\lambda_1]}, \tag{2}$$

where $e_k(\mathbf{x}) = \sum_{i_1<\ldots<i_k} x_{i_1}\cdots x_{i_k} = s_{(1^k)}(\mathbf{x})$ is the k-th elementary symmetric function.

2.4. **Dual stable Grothendieck polynomials.** The refined dual stable Grothendieck polynomials $\{g_\lambda(\mathbf{x}_a;\mathbf{z}_b)\}$ in two sets of variables $\mathbf{x}_a = (x_1,\ldots,x_b)$ and $\mathbf{z}_b = (z_1,\ldots,z_b)$, indexed by partitions λ, are defined as follows:

$$g_\lambda(\mathbf{x}_a;\mathbf{z}_b) := \sum_{\pi:\ \mathrm{sh}_1(\pi)=\lambda}\ \sum_{(i,j,k)\in\mathrm{Cor}(\pi)} x_i z_j$$

where sum runs over plane partitions $\pi \in \mathrm{PP}(a,b,c)$ of side shape $\mathrm{sh}_1(\pi) = \lambda \subseteq b\times c$.

The dual stable Grothedieck polynomials $g_\lambda(\mathbf{x}) = g_\lambda(\mathbf{x};\mathbf{z})|_{z_i\to 1}$ were introduced in [LP07] as a K-theoretic analogue of Schur functions. Their refined version $\tilde{g}(\mathbf{x};\mathbf{z}) = \mathbf{z}^\lambda g(\mathbf{x};\mathbf{z}^{-1})$ was introduced in [GGL16]. They satisfy the following determinantal formula, which is equivalent to the one proved in [Yel17].

Proposition 2.2. *For any partition λ with $\lambda_1 \le a$, we have:*

$$g_\lambda(\mathbf{x}_a;\mathbf{z}_b) = \det\left(z_{\lambda'_i}\sum_{k\ge 0} e_{j-i+k}(\mathbf{x}_a)e_k(\mathbf{z}_{\lambda'_i-1})\right)_{i,j\in[\lambda_1]}.$$

3. Paths enumeration

3.1. **Graph construction.** Let $a, b, c \in \mathbb{N}$ be fixed. Define a directed acyclic weighted graph G with vertices on the lattice $\mathbb{Z}^3$ (in X, Y, Z coordinates) as follows:

- The vertices are lattice points on the plane XY (*wall*) and the plane XZ (*floor*).
- The edges $\{e\}$ are of four types, for $i \in \mathbb{Z}$:
  $\left(\nearrow\right)$ floor *forward* edges: $(i, 0, j) \to (i, 0, j-1)$ of weight $w(e) = 1$, for $j \in [1, b]$;
  $\left(\uparrow\right)$ floor *left* edges: $(i, 0, j) \to (i-1, 0, j-1)$ of weight $w(e) = z_j$, for $j \in [1, b]$;
  $\left(\uparrow\right)$ wall *upward* edges: $(i, j-1, 0) \to (i, j, 0)$ of weight $w(e) = 1$, for $j \in [1, a]$;
  $\left(\nearrow\right)$ wall *right* edges: $(i, j, 0) \to (i+1, j+1, 0)$ of weight $w(e) = x_j$, for $j \in [1, a]$.

The directions in the edge list are specified according to Figure 2 and Figure 3. We also put c sources $\mathbf{A} = (A_1, \ldots, A_c)$ and c sinks $\mathbf{B} = (B_1, \ldots, B_c)$ for $i \in [c]$:

$$A_i = (i, 0, b), \qquad B_i = (i, a, 0),$$

i.e. each path $P : A_i \to B_j$ first 'crawls' on the floor and then 'climbs' up by the wall. Note also that $\#\{\text{left steps}\} - \#\{\text{right steps}\} = i - j$, with an equal number of left and right steps if $i = j$.

3.2. **Enumerators.** Define weighted path enumerator in the usual way:

$$w(A \to B) = \sum_{P : A \to B} \prod_{e \in P} w(e)$$

over all paths P in the lattice from the point A to the point B with steps e given as above. The weight of a path is a product of all edge weights. The following formula is then clear from the construction.

Lemma 3.1. *We have:*

$$w(A_i \to B_j) = \sum_{k \geq 0} e_k(\mathbf{z}_b) e_{j-i+k}(\mathbf{x}_a).$$

Proof. To reach B_j starting from A_i via the path P, the latter needs to have the shift $j - i$ by the coordinate x. If the path P has $k \geq 0$ extra left steps, then the shift becomes $j - i + k$. Let $P \cap OX = C = (i, 0, i-k)$. Then

$$w(A_i \to C) = e_k(\mathbf{z}_b), \qquad w(C \to B_i) = e_{j-i+k}(\mathbf{x}_a).$$

Indeed, part of P from A_i to C requires a choice of k left steps out of a possible, similarly from C to B_j requires $j - i + k$ right steps out of b available. $\square$

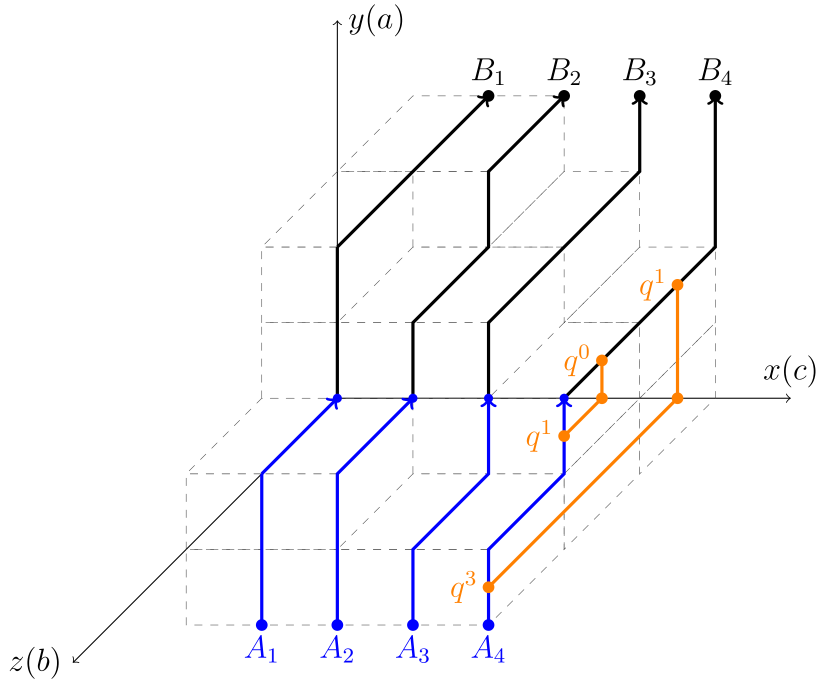


FIGURE 2. This picture can be observed in three ways: (a) boxed plane partition in $\mathrm{PP}(a, b, c)$, with dashed lines; (b) the paths $Q_i : A_i \to B_i$, where i-th one travels in the plane $x = i$, these paths define the dashed plane partition; (c) the paths $P_i : A_i \to B_i$, each going first by the 'floor' ($y = 0$ plane) and then by the 'wall' ($z = 0$ plane). The orange lines show an example of volume enumeration of (a) with edges of (b).

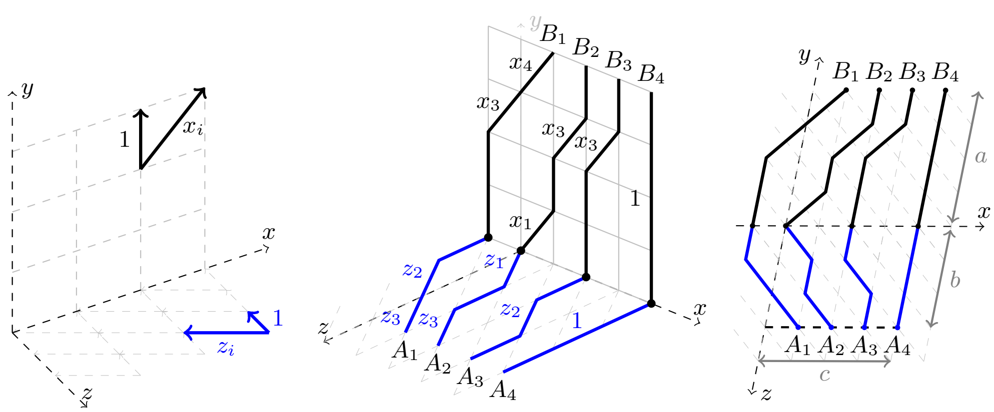


FIGURE 3. (a) A part of the lattice with typical sample steps. (b) An example of a path system for $a = 4$, $b = 3$, $c = 4$ with the path weights given by $w(P_1) = z_3z_2x_3x_4$, $w(P_2) = z_3z_1x_1x_3$, $w(P_3) = z_2x_3$, $w(P_4) = 1$. (c) A planar view of the graph G.

Similarly, define the signed weighted multi-enumerators

$$w(\mathbf{A} \to \mathbf{B}) := \sum_{\mathbf{P}:N(\mathbf{A},\mathbf{B})} \operatorname{sgn}(P) w(\mathbf{P})$$

where $N(\mathbf{A}, \mathbf{B})$ is the set of nonintersecting path systems $\mathbf{P} = (P_1, \ldots, P_c)$ from $\mathbf{A}$ to $\mathbf{B}$ (i.e. c paths with no common vertices as in Fig. 3(b)), $\operatorname{sgn}(P) := \operatorname{sgn}(\sigma)$ for $\sigma \in S_c$ if P joins A_i with $B_{\sigma(i)}$. Set

$$D(a, b, c) := \det \left(\sum_{k \geq 0} e_k(\mathbf{z}_b) e_{j-i+k}(\mathbf{x}_a) \right)_{i,j \in [c]}.$$

Lemma 3.2. *We have:*

$$D(a, b, c) = \sum_{\mathbf{P} \in N(\mathbf{A},\mathbf{B})} w(\mathbf{P}) \tag{3}$$

where $\mathbf{P} = (P_1, \ldots, P_c)$ *with* $P_i : A_i \to B_i$*.*

Proof. The proof follows by applying the Lindström–Gessel–Viennot lemma [Lin73, GV89] for the graph G, since it is in fact a planar graph. □

Lemma 3.3. *We have:*

$$\sum_{\lambda \in \mathrm{P}(\min(a,b),c)} s_\lambda(\mathbf{x}_a) s_\lambda(\mathbf{z}_b) = \sum_{\mu \in \mathrm{P}(b,c)} g_\mu(\mathbf{x}_a; \mathbf{z}_b)$$

Proof. By Lemma 3.2, non-intersecting path systems from $\mathbf{A}$ to $\mathbf{B}$ are enumerated by the determinant $D(a, b, c)$. Let us enumerate these path systems in two ways.

On the one hand, let $\mathbf{P} \in N(\mathbf{A}, \mathbf{B})$ be a non-intersecting path system. Let $\mathbf{C} = (C_1, \ldots, C_c)$ be intersection points of paths P_i with OX line for $i \in [c]$. Then $\mathbf{C}$ split the path system $\mathbf{P}$ into two path systems $\mathbf{P}_z : \mathbf{A} \to \mathbf{C}$ (floor) and $\mathbf{P}_x : \mathbf{C} \to \mathbf{B}$ (wall). Let $C_i = (i - \lambda'_i, 0, 0)$ for some vector $\lambda' = (\lambda'_1, \ldots \lambda'_c)$. Since $\mathbf{P} \in N(\mathbf{A}, \mathbf{B})$, the vector λ' is a partition. Iterating over $\mathbf{C}$, by Proposition 2.1 we may decompose the determinant as follows:

$$\begin{aligned} D(a, b, c) = &\sum_{\mathbf{C}} \det \left(w(A_i \to C_j) \right) \det \left(w(C_i \to B_j) \right) = \\ &\sum_{\lambda} \det \left(e_{\lambda'_i + j - i}(\mathbf{z}_b) \right) \det \left(e_{\lambda'_i + j - i}(\mathbf{x}_a) \right) = \sum_{\lambda} s_\lambda(\mathbf{z}_b) s_\lambda(\mathbf{x}_a), \end{aligned}$$

where the sum runs over partitions λ with $D(\lambda) \subseteq [\min(a, b)] \times [c]$. Indeed, $\lambda_1 \leq c$ since there are c paths corresponding to columns of $D(\lambda)$ diagram, and $\ell(\lambda) \leq \min(a, b)$ since the terms $s_\lambda(\mathbf{x}_a) s_\lambda(\mathbf{z}_b)$ vanish otherwise.

On the other hand, for each $i \in [c]$, let μ'_i be the maximal index for which the path P_i has $z_{\mu'_i}$ in its weight $w(P_i)$; in other words, index of the first left edge of P_i. If there is no such edge, we set $\mu'_i = 0$. Since the paths are non-intersecting, $\mu' = (\mu'_1, \ldots, \mu'_c)$ is a partition satisfying $D(\mu) \subseteq [b] \times [c]$. Indeed, it has at most c columns (as there are c paths) and each $\mu'_i \le b$ by construction.

Denote by $A'_i = (i-1, 0, \mu'_i - 1)$ the point of P_i after passing this edge for $i \in [\mu_1]$. Summing up over all possible $\mathbf{A}' = (A'_1, \ldots, A'_c)$ (or equivalently, over μ') and using Proposition 2.2 we get:

$$\begin{aligned} D(a,b,c) &= \sum_{\mathbf{A}'} \det\left(z_{\mu'_i} w(A'_i \to B_j)\right) \\ &= \sum_{\mu'} \det\left(z_{\mu'_i} \sum_{k \ge 0} e_k(\mathbf{z}_{\mu'_i - 1}) e_{j-i+k}(\mathbf{x_b})\right) = \sum_{\mu'} g_\mu(\mathbf{x}_a; \mathbf{z}_b). \end{aligned}$$

This completes the proof. $\square$

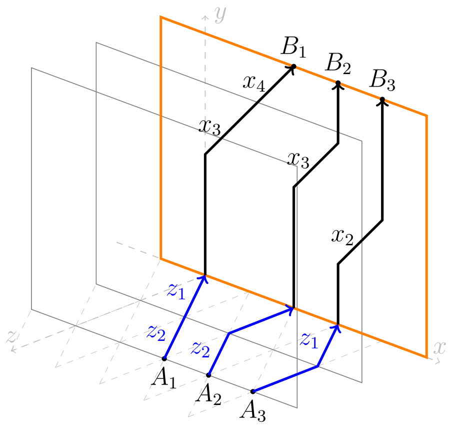
Remark 2. This identity was shown in [MS20] by a probabilistic argument using the connection of dual stable Grothendieck polynomials with last passage percolation model from [Yel20] and the Schur measure.

Remark 3. We now have a complete 'triangle' of formulas and relations between them (for $c \to \infty$):

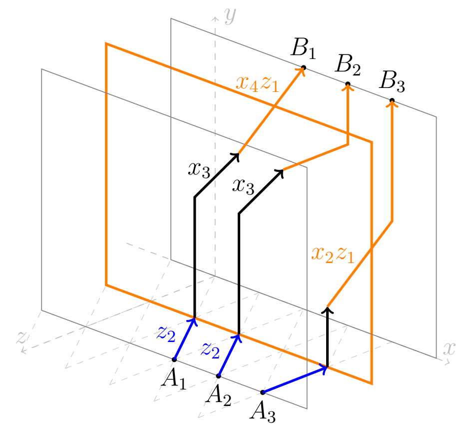
- The RSK correspondence (see e.g. [Sta99, Ch. 7]) shows
$$\sum_{\ell(\lambda) \le \min(a,b)} s_\lambda(\mathbf{x}_a) s_\lambda(\mathbf{z}_b) = \prod_{i=1}^{a} \prod_{j=1}^{b} \frac{1}{1 - x_i z_j}$$
- The bijective map $\Phi : \mathrm{PP}(a, b, \infty) \to \{(a_{i,j})_{i \in [a], j \in [b]} : a_{i,j} \in \mathbb{N}\}$ (see [Yel21a, Yel21b, AY23]) shows
$$\sum_{\ell(\lambda) \le b} g_\lambda(\mathbf{x}_a; \mathbf{z}_b) = \prod_{i=1}^{a} \prod_{j=1}^{b} \frac{1}{1 - x_i z_j}$$
- Lemma 3.3 directly shows
$$\sum_{\ell(\lambda) \le \min(a,b)} s_\lambda(\mathbf{x}_a) s_\lambda(\mathbf{z}_b) = \sum_{\ell(\lambda) \le b} g_\lambda(\mathbf{x}_a; \mathbf{z}_b).$$

Remark 4. Our proof can be converted to a direct bijection using 'jeu de taquin' like operations on paths described in [AY22]. In particular, in [AY22] we defined the operations $\{\mathrm{slide}_k\}_{k \ge 0}$,[1] describing procedure which transforms 3d non-intersecting path system $\mathbf{P}^{(i)} : \mathbf{A} \to \mathbf{B}$ to 3d non-intersecting path system $\mathbf{P}^{(i+1)} : \mathbf{A} \to \mathbf{B}$. Each path $P_j^{(i)}$ travels as follows: first crawls by the floor $y = 0$, then climbs by the wall $z = k$, and then travels on the plane $x = i$. Then starting from $\mathbf{P}^{(0)} := \mathbf{P} \in N(\mathbf{A}, \mathbf{B})$ and applying consequently $\mathrm{slide}_{b-1} \circ \ldots \circ \mathrm{slide}_0$ to $\mathbf{P}$ we can construct a new path system $\mathbf{P}' : \mathbf{A} \to \mathbf{B}$ with each P'_i travelling in $x = i$ plane, with steps $(0, 1, 0)$, $(0, 0, 1)$ and $(0, 1, 1)$, where the latter step

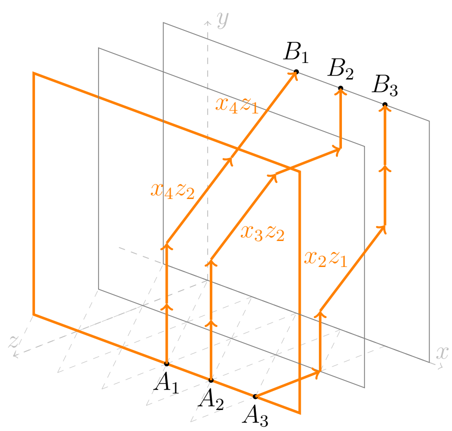
[1] As their definitions are somewhat technical and long, we do not reproduce them here.



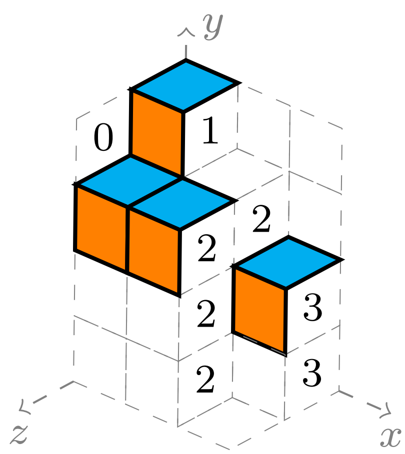
(A) Initial non-intersecting path system $\mathbf{P}$.



(B) Operation slide_0 applied: path P_i travels in $x = i$ in its tail (orange).



(C) Operation slide_1 applied: floor and wall parts of P_i are combined into path travelling in $x = i$ plane.



(D) The resulting plane partition.

FIGURE 4. From a non-intersecting path system to plane partition.

type corresponds to corners of the new plane partition, see Figure 4 (cf. [AY22, Fig. 7]). This defines direct a weight preserving bijection between plane partitions enumerated by corners in $g_\lambda(\mathbf{x};\mathbf{z})$ and non-intersecting path systems in $D(a, b, c)$.

4. Equidistribution theorem

In this section we show implications of Section 3.

Theorem 4.1 (= Theorem 1.1)**.** *We have*

$$\sum_{\pi \in \mathrm{PP}(a,b,c)} q^{|\pi|} t^{\mathrm{tr}(\pi)} = \sum_{\pi \in \mathrm{PP}(a,b,c)} q^{|\pi|_{ch}} t^{\mathrm{cor}(\pi)} \tag{4}$$

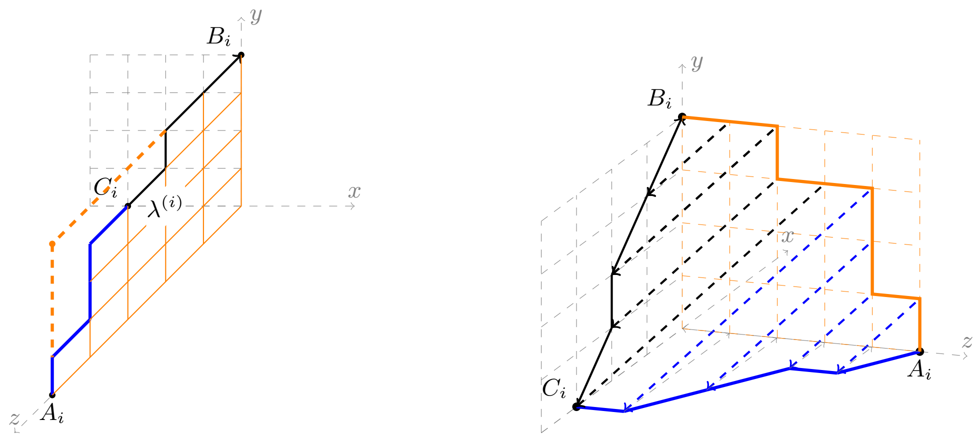


FIGURE 5. The path P_i from $A_i = (i, 0, c)$ to $B_i = (i, b, 0)$ intersecting OX at point $C_i = (i - \ell_i, 0, 0)$. (a) The Path P_i and Q_i coincide under this view angle. (b) The path Q_i (orange) travels on the plane $x = i$ and bounds the partition $\lambda^{(i)} = (5, 4, 4, 2)$, the projection lines are displayed to see the correspondence between the paths P_i and Q_i.

Proof. We refer to the Figure 2 for visualization. In Lemma 3.3 set $x_i = q^i, z_i = tq^{i-1}$ to obtain:

$$\sum_{\lambda \subseteq \min(a,b) \times c} s_\lambda(q, q^2, \ldots, q^a) s_\lambda(t, tq, \ldots, tq^{b-1}) = \sum_{\lambda \subseteq b \times c} g_\lambda(q, \ldots, q^a; t, \ldots, tq^{b-1}).$$

On the RHS of (4) we already obtain enumeration of plane partitions by corner-hook volume:

$$\begin{aligned} g_\lambda(q, \ldots, q^a; t, \ldots, tq^{b-1}) &= \sum_{\pi \in \mathrm{PP}(a,b,c):\ \mathrm{sh}_1(\pi) = \lambda} \ \prod_{(i,j,k) \in \mathrm{Cor}(\pi)} q^j \cdot tq^{i-1} \\ &= \sum_{\pi \in \mathrm{PP}(a,b,c):\ \mathrm{sh}_1(\pi) = \lambda} t^{\mathrm{cor}(\pi)} q^{|\pi|_{ch}}. \end{aligned}$$

Hence, the RHS enumerates plane partitions by corners in the box $\mathrm{PP}(a, b, c)$.

On the LHS, we interpret each term as a non-intersecting path system $\mathbf{P}$. View each path $P_i \in \mathbf{P}$ in other way. Project path $P_i : A_i \to B_i$ to the plane $x = i$ along the vector $(1, 1, 1)$ to obtain the path $Q_i : A_i \to B_i$ which uses $(0, 1, 0)$ and $(0, 0, 1)$ steps and travels within the plane $x = i$. It is easy to see (visually) that this is a bijection. See Figure 5 for a visual explanation.

The path Q_i is then the boundary of some partition $\lambda^{(i)} \subseteq a \times b$ (of its diagram), drawn in the plane $x = i$ (in French notation). Since the paths are non-intersecting, $\pi := (\lambda^{(1)} \supseteq \ldots \supseteq \lambda^{(c)})$ is a plane partition, where $\pi_i := \lambda^{(i)}$ is its i-th row. This defines a bijection between non-intersecting path systems $\mathbf{P}$ enumerated by $D(a, b, c)$ and plane partitions $\pi \in \mathrm{PP}(a, b, c)$.

Let us show that the contribution of $w(\mathbf{P})$ matches the required term $t^{\mathrm{tr}(\pi)}q^{|\pi|}$. For each $i \in [c]$ write the partition $\lambda^{(i)} = (a_1, \ldots, a_{\ell_i}|b_1 \ldots, b_{\ell_i})$ in Frobenius notation, where $\ell_i := d(\lambda^{(i)})$. Then the path P_i has $2\ell_i$ edges, and moreover, the hook (a_j, b_j) corresponds to the pair of edges of P_i: floor left edge z_{b_j} and wall right edge x_{a_j} (see orange lines in Figure 2). Thus, for each $i \in [c]$ we have

$$w(P_i) = \prod_{j=1}^{\ell_i} z_{b_j} x_{a_j} = \prod_{j=1}^{\ell_i} q^{b_j} \cdot tq^{a_j - 1} = q^{\sum (a_j + b_j - 1)} t^{\ell_i} = q^{|\lambda^{(i)}|} t^{\ell_i}$$

and consequently

$$w(\mathbf{P}) = \prod_{i=1}^{a} w(P_i) = q^{\sum |\lambda^{(i)}|}\, t^{\sum \ell_i} = q^{|\pi|} t^{\mathrm{tr}(\pi)},$$

since $\mathrm{tr}(\pi) = \sum_{i=1}^{c} d(\lambda^{(i)})$. This completes the proof. □

As a byproduct of the above discussions we obtain the following corollaries.

Corollary 4.2 (Fixed trace generating function)**.** *For a partition* $\lambda \in \mathrm{P}(\min(a, b), c)$ *we have:*

$$s_\lambda(q, q^2, \ldots, q^a) s_\lambda(t, tq, \ldots, tq^{b-1}) = \sum_{\pi \in \mathrm{PP}(a,b,c):\ \mathrm{tr}(\pi) = \lambda} q^{|\pi|} t^{\mathrm{tr}(\pi)}$$

Corollary 4.3 (Boxed PP volume enumeration)**.** *For a partition* $\lambda \in \mathrm{P}(\min(a, b), c)$ *we have:*

$$D(a, b, c)|_{x_i \to q^i, z_j \to q^{j-1}} = \sum_{\pi \in \mathrm{PP}(a,b,c)} q^{|\pi|}$$

Remark 5. Notably, the latter corollary is obtained without the use of the RSK correspondence. Indeed, $D(a, b, c)$ enumerates non-intersecting paths by LGV lemma (Lemma 3.2) and the interpretation of $w(\mathbf{P})$ for $\mathbf{P} \in N(\mathbf{A}, \mathbf{B})$ after specialization is direct and visual.

Remark 6. Since the volume and the corner-hook volume are equidistributed, it is natural to ask if they also have *symmetric joint distribution*, i.e. whether

$$\sum_{\pi} q^{|\pi|} t^{|\pi|_{ch}} = \sum_{\pi} q^{|\pi|_{ch}} t^{|\pi|}$$

holds. While (as we checked) this equality does *not* hold over all boxed plane partitions, it seems to hold for sums over $\pi \in \mathrm{PP}(2, 2, c)$, which would be interesting to prove.

5. Partition corner hook area

For a partition λ, recall that $|\lambda| = |D(\lambda)|$ is its area, $d(\lambda) := \max_{\lambda_k \geq k} k$ is the size of its Durfee square, and $h_\lambda(i, j) := (\lambda_i - i) + (\lambda'_j - j) + 1$ denotes the *hook length* of the cell (i, j). We clearly have

$$|\lambda| = \sum_{i=1}^{d(\lambda)} h_\lambda(i, i).$$

Denote by $p(n,k)$ the number of partitions λ with $|\lambda| = n$ and $d(\lambda) = k$ and set $p(n) = \sum_k p(n,k)$ the total number of partitions of n.

Definition 5.1. Let λ be a partition. The cell $(i,j) \in D(\lambda)$ is called a *corner*, if $(i+1,j)$ and $(i,j+1)$ are not in $D(\lambda)$. The set of corners is denoted by $\mathrm{Cor}(\lambda)$ and its size is $\mathrm{cor}(\lambda) := |\mathrm{Cor}(\lambda)|$. By $\mathrm{ch}(i,j) := i+j-1$ we denote a *cohook length* of the cell (i,j). Define the *cohook area* $|\lambda|_c$ as follows:

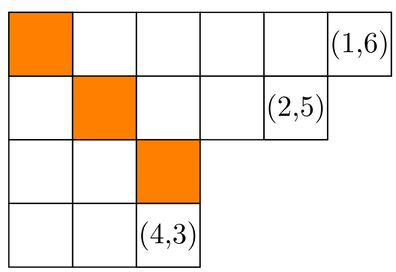
$$|\lambda|_c := \sum_{(i,j)\in \mathrm{Cor}(\lambda)} \mathrm{ch}(i,j).$$

For example, the partition $\lambda = (21)$ with $|\lambda| = 3$ has two corners $(1,2)$ and $(2,1)$, and hence, $|\lambda|_c = 2+2 = 4$. Denote by $q(n,k)$ the number of partitions with $|\lambda|_c = n$ and $\mathrm{cor}(\lambda) = k$ and set $q(n) = \sum_k q(n,k)$.

Since any partition λ is uniquely determined by its corners, let us introduce the *corner notation*. By $[a_1,\ldots,a_k|b_1,\ldots,b_k]$ we denote a partition with corners at $(a_1,b_1),\ldots,(a_k,b_k)$, so that

$$a_1 > \ldots > a_k \geq 1 \quad \text{and} \quad 1 \leq b_1 < \ldots < b_k,$$

since no two corners can occupy the same row or column. For example,

$\lambda = [4,2,1|3,5,6] =$  $= (6,4,1|4,3,2),$

with $|\lambda|_c = 19$ and $|\lambda| = 17$.

Theorem 5.2. *For any $n,k \geq 1$ we have:*

$$p(n,k) = q(n,k),$$

i.e. the paired partition statistics $(|\cdot|, d(\cdot))$ and $(|\cdot|_c, \mathrm{cor}(\cdot))$ are equidistributed.

Proof. Let $\lambda = (a_1,\ldots,a_k|b_1,\ldots,b_k)$ be a partition of n written in Frobenius notation. Consider the following map

$$\Phi : \lambda = (a_1,\ldots,a_k|b_1,\ldots,b_k) \mapsto [a_1,\ldots,a_k|b_k,\ldots,b_1] =: \mu$$

from Frobenius to corner notation. Then we have $|\lambda| = \sum a_i + b_i - 1 = \sum a_i + b_{n+1-i} - 1 = |\mu|_c$. This shows that Φ is in fact a bijection

$$\Phi : \{\lambda : |\lambda| = n, d(\lambda) = k\} \to \{\lambda : |\lambda|_c = n, \mathrm{cor}(\lambda) = k\}$$

which implies the claim. □

The following corollary is immediate.

Corollary 5.3. *For any $n \geq 1$, we have: $p(n) = q(n)$.*

Analogous to Theorem 1.1, we present the boxed version.

Theorem 5.4 (= Theorem 1.2)**.** *We have the following equidistribution of paired statistics $(|\cdot|, d(\cdot))$ and $(|\cdot|_c, \mathrm{cor}(\cdot))$ within the box $[m] \times [n]$:*

$$\sum_{\lambda \in \mathrm{P}(m,n)} q^{|\lambda|} t^{d(\lambda)} = \sum_{\lambda \in \mathrm{P}(m,n)} q^{|\lambda|_c} t^{\mathrm{cor}(\lambda)}.$$

Proof. We are enough to verify that $D(\Phi(\lambda)) \subseteq [m] \times [n]$ whenever $D(\lambda) \subseteq [m] \times [n]$. Apply the map Φ to such λ written in Frobenius notation as $(a_1, \ldots, a_k | b_1, \ldots, b_k)$. Then $\{a_i\}$ is a strictly decreasing subsequence of $[n]$, and similarly $\{b_i\}$ of $[m]$. Let $\mu := \Phi(\lambda) = [a_1, \ldots, a_k | b_k, \ldots, b_1]$ in corner notation, then each $a_i \in [n]$ and $b_i \in [m]$ ensuring that $D(\mu) \subseteq [m] \times [n]$. The reverse direction is similar. $\square$

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