Everything we know about higher-dimensional partitions

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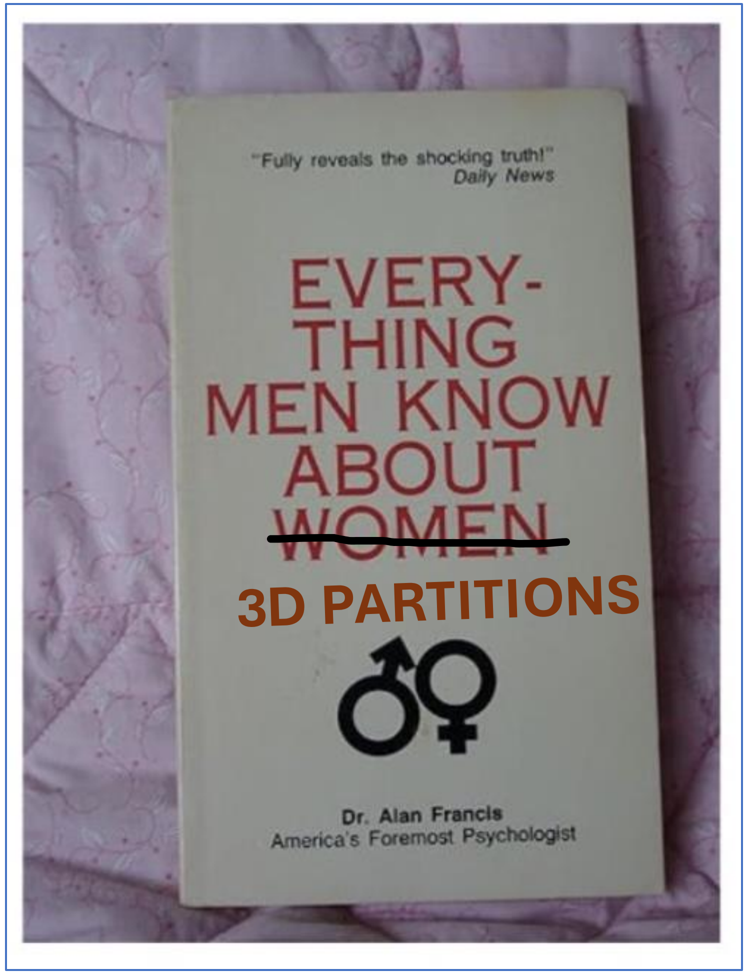
Based on joint work with Alimzhan Amanov

IPAM seminar (GSI program)

May 1, 2024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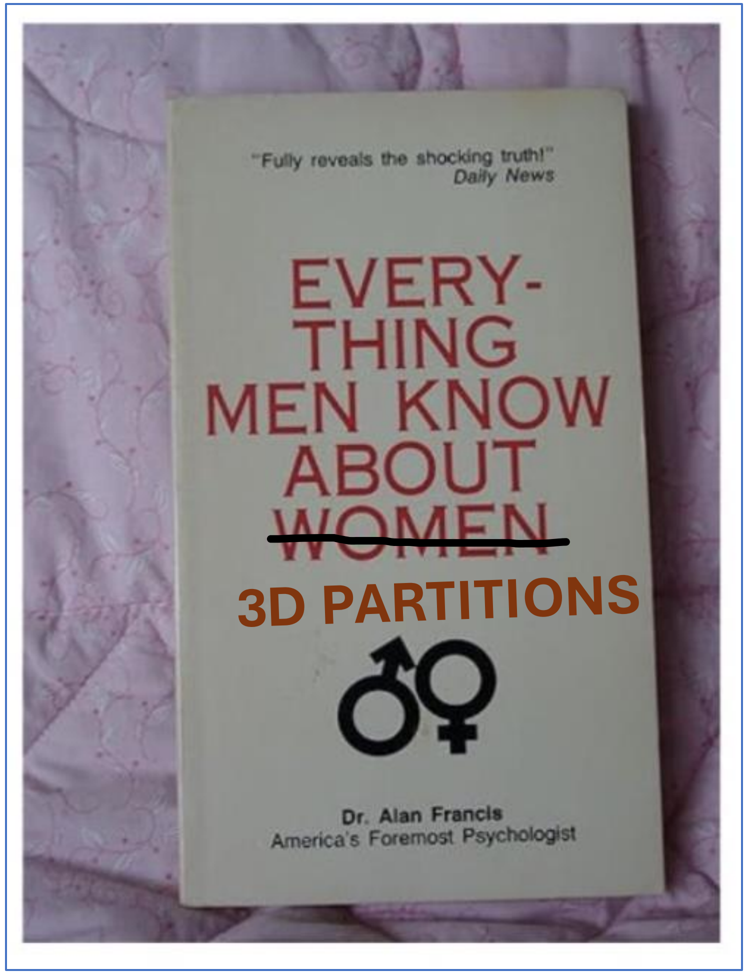
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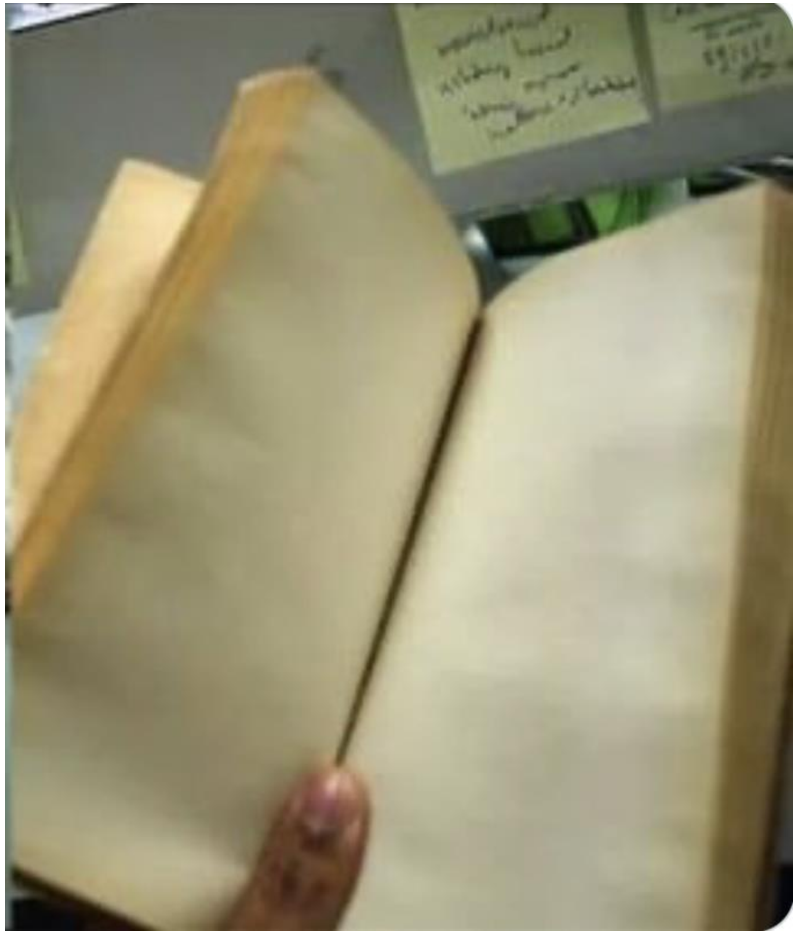

Q. How to generalize integer partitions in higher dimensions?

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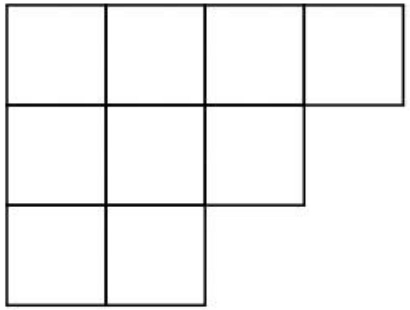
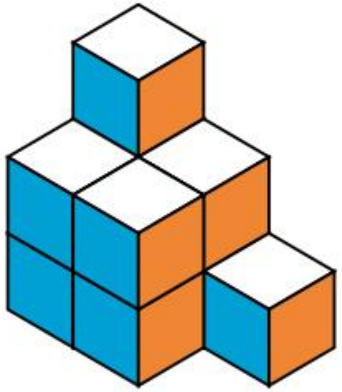
A: Easy

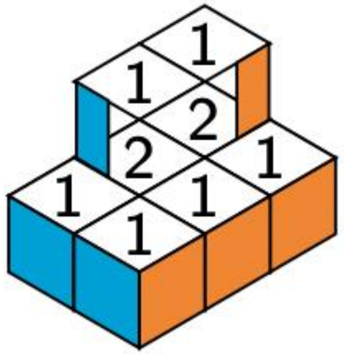
Usual integer partitions (1-d) (λ_i)



Plane partitions (2-d) (π_{ij})



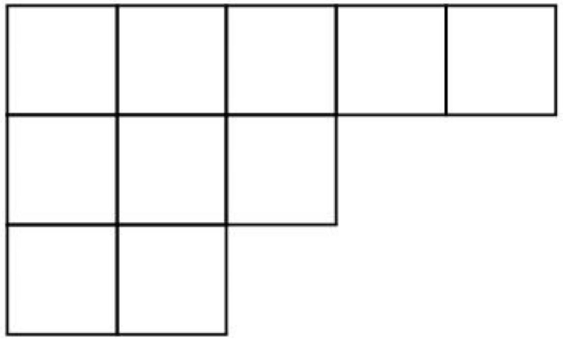
Solid partitions (3-d) (π_{ijk})



Integer partitions (1-d)

$$\lambda = (\lambda_1 \geq \cdots \geq \lambda_\ell) \qquad |\lambda| = \sum \lambda_i$$

$$\lambda = (5, 3, 2)$$

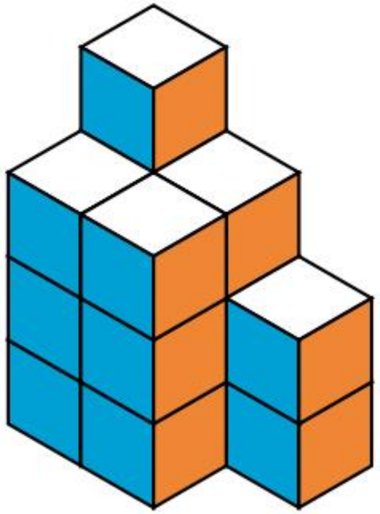
Diagram: 

Plane partitions (2-d)

$$\pi = (\pi_{ij}) \qquad \pi_{ij} \geq \pi_{i+1\,j}, \pi_{i\,j+1} \qquad |\pi| = \sum \pi_{ij}$$

$\pi =$

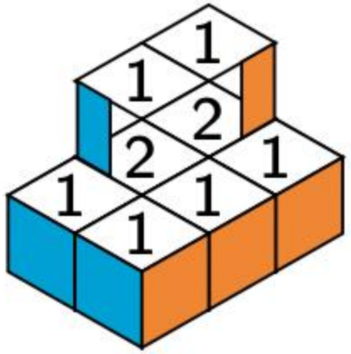
4	3	2
3	3	

Diagram: 

d-dimensional partitions

$\mathbb{N}$-tensors $(\pi_{i_1 \dots i_d})$ $\qquad$ $\pi_{i_1 \dots i_d} \geq \pi_{j_1 \dots j_d}$ for $i_1 \geq j_1, \dots, i_d \geq j_d$

volume $|\pi| = \sum_{i_1,\dots,i_d} \pi_{i_1,\dots,i_d}$

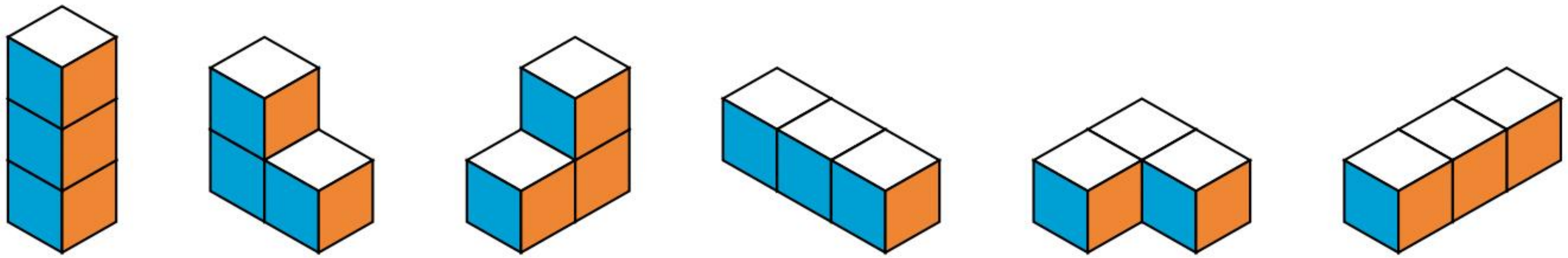
3-d partition: 

with 4-d diagram of volume 10.

Enumeration and generating functions

$p_d(n)$ number of d-dimensional partitions of volume n

$p_2(3) = 6$



1-d review

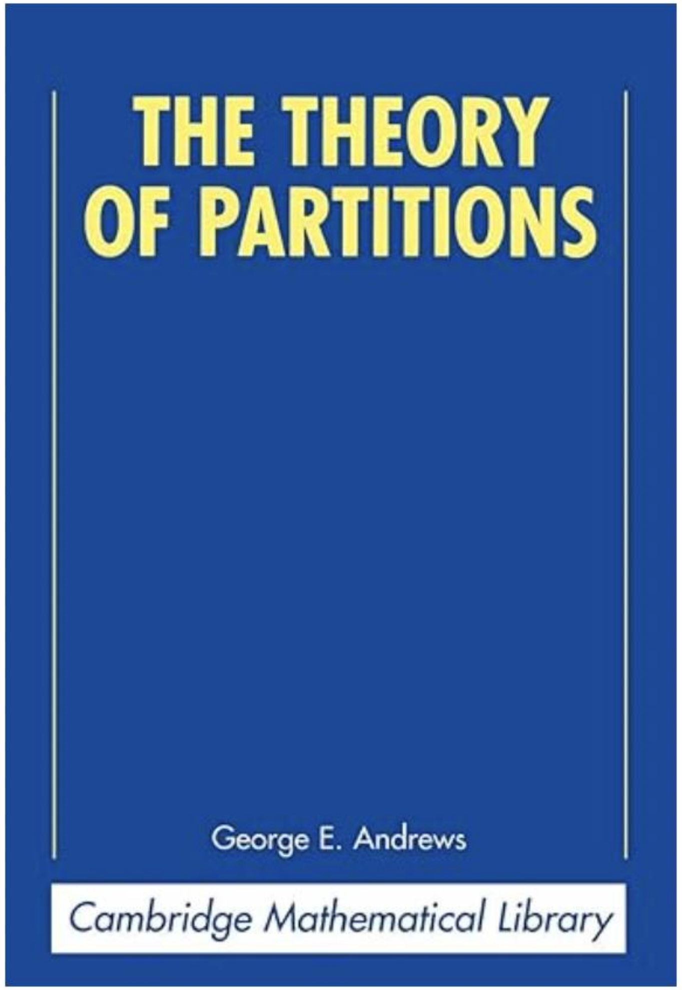
Theorem. (Euler)

$$y(t) = \sum_{\lambda} t^{|\lambda|} = \sum_{n} p(n) t^n = 1+t+2t^2+3t^3+5t^4+\cdots = \prod_{n=1}^{\infty} \frac{1}{1-t^n}$$

Dedekind eta function $\eta(z) = t^{1/24}/y(t)$ $(t = e^{2\pi i z})$ is a *modular form*, i.e. has SL_2 translation $\eta(z+1) = t^{1/24}\eta(z)$ and $\eta(-1/z) = \sqrt{z/i}\cdot\eta(z)$.

(Classics) $y(t)$ is a solution to *algebraic differential equation*

1-d some refs


THE THEORY
OF PARTITIONS
George E. Andrews
Cambridge Mathematical Library


Home > The Ramanujan Journal > Article
Partition bijections, a survey
Published: August 2006
Volume 12, pages 5–75, (2006)
Cite this article
Igor Pak

2-d review

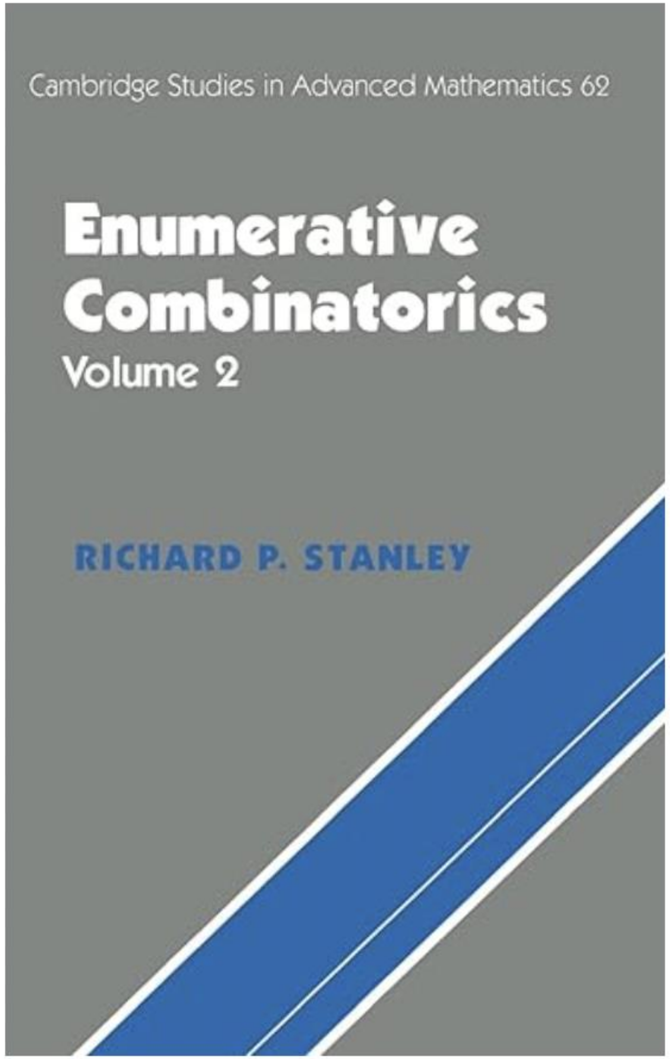
Theorem. (MacMahon 1890)

$$\sum_{\pi \text{ plane partitions}} t^{|\pi|} = \sum_n p_2(n) t^n = \prod_{n=1}^{\infty} \frac{1}{(1-t^n)^n}$$

- Proof uses RSK, Schur functions
- There are many known refinements of this generating function

$$\sum_{\pi \text{ in } [a]\times[b]\times[c]} t^{|\pi|} = \prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{1-t^{i+j+k-1}}{1-t^{i+j+k-2}}$$

2-d some refs



PLANE PARTITIONS IN THE WORK OF RICHARD STANLEY AND HIS SCHOOL

C. KRATTENTHALER

ABSTRACT. These notes provide a survey of the theory of plane partitions, seen through the glasses of the work of Richard Stanley and his school.

1. INTRODUCTION

Plane partitions were introduced to (combinatorial) mathematics by Major Percy Alexander MacMahon [71] around 1900. What he had in mind was a planar analogue of a(n integer) partition.[1]

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-I can do p(200)

The hero

The hero

MacMahon's conjecture (1916).
The generating function for d-dim partitions is

$$\sum_{\pi \ d\text{-dim partitions}} t^{|\pi|} = \sum_n p_d(n) t^n \stackrel{?}{=} \prod_{n=1}^{\infty} \frac{1}{(1-t^n)^{\binom{n+d-2}{d-1}}}$$

Note: true for $d = 1, 2$

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Theorem. (Atkin-Bratley-Macdonald-McKay 1967)
Conjecture is **false** for $d \geq 3$ and $n \geq 6$.

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D. Knuth ('70) *"The problem of enumerating* **three-dimensional ("solid") partitions has never been resolved**, *... and Part VII of MacMahon's classic Memoir* **never appeared**. *No constructive proof of MacMahon's formula for the two-dimensional case was known until 1969."*

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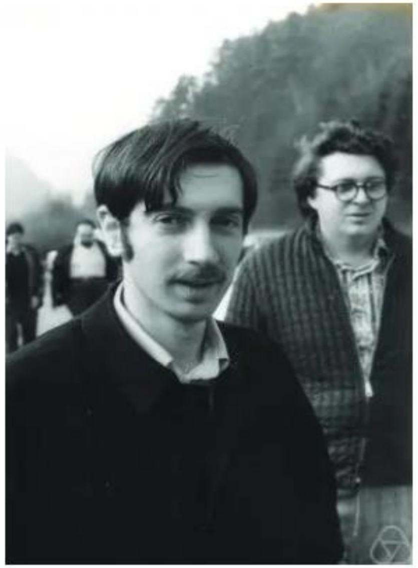

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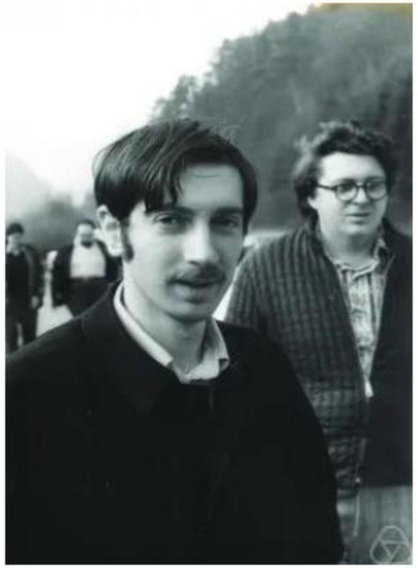

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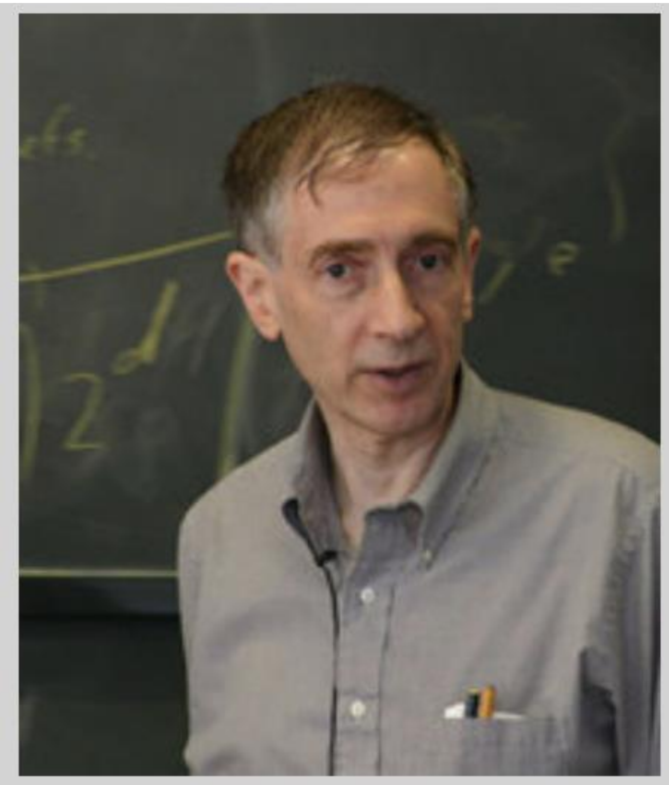
R. Stanley ('99, EC2): *" It now seems obvious to define* d*-dimensional partitions for any* $d \geq 1$*. However, almost* **nothing significant is known** *for* $d \geq 3$*. "*

ecajournal.haifa.ac.il Enumerative Combinatorics and Applications ECA **1** (2021) Article #S3I1

Interview with Richard P. Stanley

Toufik Mansour



Mansour: Were there specific problems that made you first interested in combinatorics?

Stanley: Perhaps the next such problem was the enumeration of solid (3-dimensional partitions), generalizing MacMahon's famous enumeration of plane partitions. I never made significant progress (and most likely the problem is intractable), but it did lead me to the theory of P-partitions, the subject of my Ph.D. thesis.

Q (or hope): Maybe MacMahon wasn't that wrong?

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A: Maybe

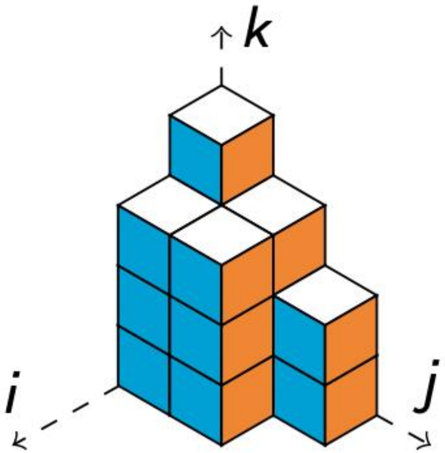
A 'corrected' version

Theorem. (Amanov–Y. 2020) There is a statistic $|\cdot|_{ch}$ on d-dim partitions called **corner-hook volume** such that

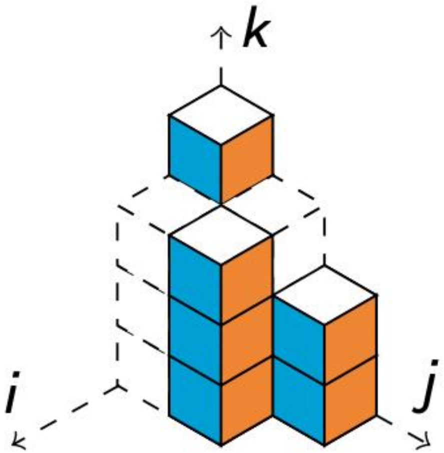
$$\sum_{\pi \text{ } d\text{-dim partitions}} t^{|\pi|_{ch}} = \prod_{n=1}^{\infty} \frac{1}{(1-t^n)^{\binom{n+d-2}{d-1}}}$$

Corner-hook volume statistic

Diagram $D(\pi) \in \mathbb{Z}^{d+1}$ of d-dim partition π.



Corners:



$$\mathrm{Cor}(\pi) := \{\mathrm{i} \in \mathbb{Z}_+^{d+1} : \mathrm{i} \in D(\pi),\ \mathrm{i} + \mathrm{e}_\ell \notin D(\pi) \text{ for all } \ell \in [d]\}.$$

$$|\pi|_{ch} = \sum_{(i_1,\ldots,i_{d+1}) \in \mathrm{Cor}(\pi)} (i_1 + \ldots + i_d - d + 1)$$

$|\pi|_{ch} = |\pi|$ for $d = 1$

$|\pi|_{ch} \neq |\pi|$ for $d \geq 2$

$$\begin{aligned}\mathrm{Cor}(\pi) &= \{(i,j,k) \in D(\pi) : (i+1,j,k),(i,j+1,k) \notin D(\pi)\} \\ &= \{(1,1,4),(1,3,1),(1,3,2),(2,2,1),(2,2,2),(2,2,3)\}\end{aligned}$$

$$|\pi|_{ch} = (1+1-1)+(1+3-1)+(1+3-1)+(2+2-1)+(2+2-1)+(2+2-1) = 16.$$

More generating functions [AY '20]

Theorem 5.2. *Let $\rho \subset \mathbb{Z}_+^d$ be a fixed shape of a d-dimensional partition. We have the following generating functions:*

$$\sum_{\pi \in \mathcal{P}^{(d)},\, \mathrm{sh}(\pi) \subseteq \rho} t^{\mathrm{cor}(\pi)} q^{|\pi|_{ch}} = \prod_{(i_1,\ldots,i_d) \in \rho} \left(1 - tq^{i_1+\cdots+i_d-d+1}\right)^{-1},$$

Corollary 5.3 (Boxed version). *We have*

$$\sum_{\pi \in \mathcal{P}(n_1,\ldots,n_d,\infty)} t^{\mathrm{cor}(\pi)} q^{|\pi|_{ch}} = \prod_{i_1=1}^{n_1} \cdots \prod_{i_d=1}^{n_d} \left(1 - tq^{i_1+\cdots+i_d-d+1}\right)^{-1}.$$

Corollary 5.4 (Full generating function). *We have*

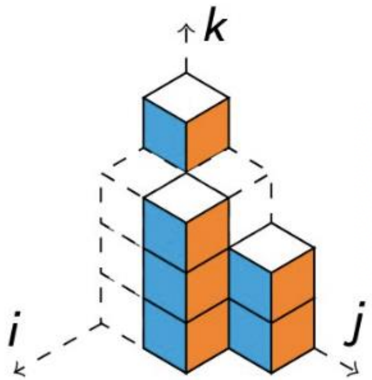
$$\sum_{\pi \in \mathcal{P}^{(d)}} t^{\mathrm{cor}(\pi)} q^{|\pi|_{ch}} = \prod_{n \geq 1} (1 - tq^n)^{-\binom{n+d-2}{d-1}}.$$

Proofs via "corner projection" bijection

Let $\mathcal{M}^{(d)}$ be the set of d-dimensional $\mathbb{N}$-hypermatrices and $\mathcal{P}^{(d)}$ be the set of d-dimensional partitions.

Consider the *corner projection map* $\varphi : \mathcal{P}^{(d)} \to \mathcal{M}^{(d)}$ given by $\pi \mapsto (a_{\mathbf{i}})$, where

$$a_{\mathbf{i}} = |\{i_{d+1} : (\mathbf{i}, i_{d+1}) \in \mathrm{Cor}(\pi)\}|, \quad \mathbf{i} \in \mathbb{Z}_+^d.$$



$$(a_{ij}) = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & 0 \end{pmatrix}$$

- Its inverse can be viewed as directed last passage percolation map
- Also related to Stanley's transfer map between poset and order polytopes
- For $d = 2$ in [Y. 19] with applications coming from dual Grothendieck polynomials

3d Grothendieck polynomials [AY '20]

$$g_\rho(\mathbf{x};\mathbf{y};\mathbf{z}) := \sum_{\pi\,:\,\mathrm{sh}_1(\pi)=\rho}\ \prod_{(i,j,k,\ell)\in\mathrm{Cor}(\pi)} x_i y_j z_k$$

$g_\rho(\mathbf{x};\mathbf{y};\mathbf{z})$ indexed by plane partitions ρ.

solid partitions $\pi \in \mathcal{P}(n_1, n_2, n_3, n_4)$

$$\sum_{\rho\in\mathcal{P}(n_2,n_3,\infty)} g_\rho(\mathbf{x};\mathbf{y};\mathbf{z}) = \prod_{i=1}^{n_1}\prod_{j=1}^{n_2}\prod_{k=1}^{n_3}\left(1 - x_i y_j z_k\right)^{-1}$$

$$g_{[n_2]\times[n_3]\times[n_4]}(1^{n_1+1}) = |\mathcal{P}(n_1, n_2, n_3, n_4)|$$

the number of solid partitions inside the box $[n_1]\times[n_2]\times[n_3]\times[n_4]$

- Quasisymmetric in **x**
- Generalize symmetric (2-d) dual Grothendieck polynomials of [Lam-Pylyavskyy '07]
- Determine probability for directed 3d last passage percolation with geom weights [AY '20]

Asymptotics

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$\log p_1(n) \sim c_1 n^{1/2}, \qquad c_1 = 2\zeta(2)^{1/2} \qquad$ (Hardy-Ramanujan)

$\log p_2(n) \sim c_2 n^{2/3}, \qquad c_2 = 3/2^{2/3}\zeta(3)^{1/3} \qquad$ (Wright, '31)

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Q: Asymptotics of $p_d(n)$?

Open problem. Prove limit exists and find it

$$\lim_{n\to\infty} \frac{\log p_d(n)}{n^{d/(d+1)}} = c_d = ?$$

Asymptotics of MacMahon's numbers

MacMahon's numbers $m_d(n)$:

$$\sum_n m_d(n) t^n = \prod_{n=1}^{\infty} \frac{1}{(1-t^n)^{\binom{n+d-2}{d-1}}}$$

$$\log m_d(n) \sim \gamma_d n^{d/(d+1)}, \qquad \gamma_d = \frac{d+1}{d^{d/(d+1)}} \zeta(d+1)^{1/(d+1)}$$

Simulations by physicists

Conjecture 1. (Mustonen–Rajesh, J. Phys. A '03)

$$\log p_3(n) \sim \log m_3(n) \sim 1.78..n^{3/4}$$

Conjecture 2. (Balakrishnan-Govindarajan-Prabhakar, J. Phys. A '12)

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Simulation 3. (Destainville-Govindarajan, J. Stat. Phys. '15)

$$\log p_3(n) \sim 1.82..n^{3/4}$$

What's known

Theorem. (Bhatia–Prasad–Arora, '97) $\log p_d(n) = \Theta(n^{d/(d+1)})$

What's known

Q: Explicit lower/upper bounds?

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Theorem. (Y. '23)

$$\frac{(d+1)}{(d+1)^{1/(d+1)}} (\log 2 + \epsilon_d) \leq \frac{\log p_d(n)}{n^{d/(d+1)}} \leq (d+1)\zeta(d+1)^{1/(d+1)} + o(1)$$

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- Lower bound #partitions pyramid
- Upper bound via corner-hook vol

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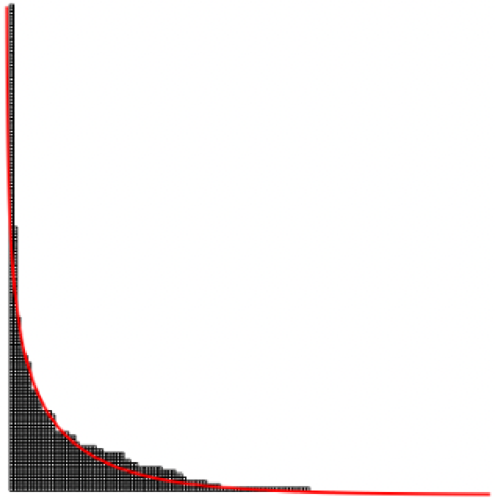
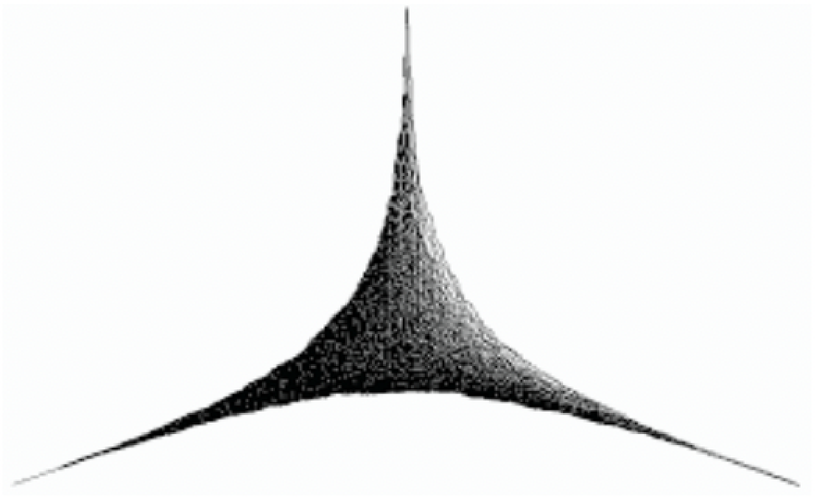
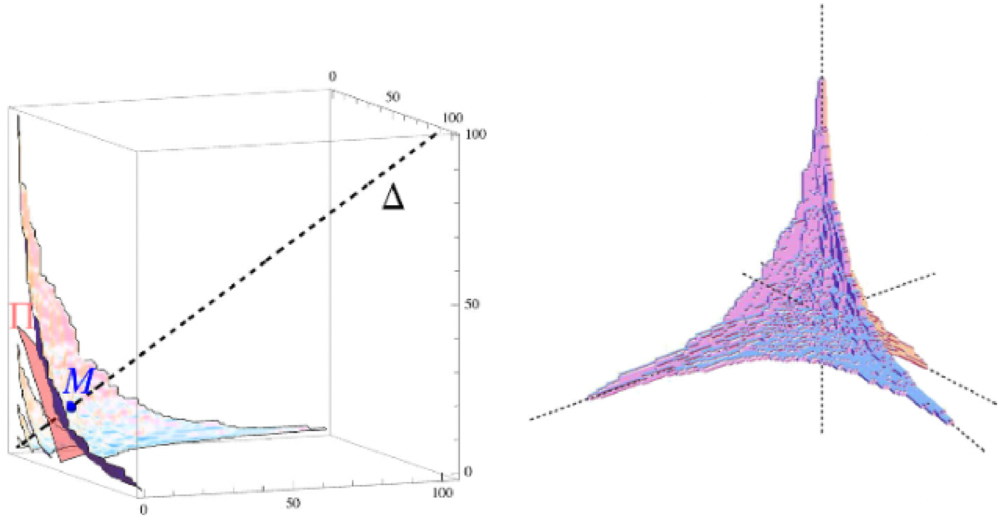
Theorem. (Oganesyan, '23) For sufficiently large n, $\frac{\log p_d(n)}{n^{d/(d+1)}} < 4200$

Limit shapes of random partitions





$d = 1$ Vershik $\qquad$ $d = 2$ Okounkov–Reshetikhin $\qquad$ $d \geq 3$???



$d = 3$ Destainville–Govindarajan simulation

Boxed d-dimensional partitions

Let $P_d(n)$ be the number of d-dimensional partitions with diagram inside the box $[n]^{d+1}$

$$P_1(n) = \binom{2n}{n}, \quad P_2(n) = \prod_{i,j,k=1}^{n} \frac{i+j+k-1}{i+j+k-2}$$

Theorem. (Moshkowitz–Shapira, 2014)

$$\frac{2}{3\sqrt{d+1}} \leq \frac{\log_2 P_d(n)}{n^d} \leq 2$$

Problem. Show as $n \to \infty$ limit exists and find it.

– Related to Ramsey theory and the number of poset antichains

– Some more and $d \to \infty$ studied in [Pohoata–Zaharov '21, Park–Sarantis–Tetali '23, Falgas-Ravry–Räty-Tomon '23]

Partitions inside pyramid

Let $A_d(n)$ be the number of d-dimensional partitions with diagram inside the simplex $x_1 + \ldots + x_{d+1} \leq n$

Theorem. (Y. '23)

$$1 \leq \frac{\log_2 A_d(n)}{\binom{n-1}{d}} \leq 2$$

Problem. Show as $n \to \infty$ limit exists and find it.

Complementary boxed partitions

Another recent explicit generating function

[F. Schreirer-Aigner 2023]

Theorem 1.1. *Let* $\mathbf{x} = (x_1, \ldots, x_{d+1})$, $\mathbf{n} = (n_1, \ldots, n_{d+1}) \in \mathbb{N}_{>0}^{d+1}$ *and denote by* $\mathrm{FCP}(\mathbf{n})$ *the set of fully complementary partitions inside a* $(2n_1, \ldots, 2n_{d+1})$*-box. Then*

$$\sum_{\mathbf{n}\in\mathbb{N}^{d+1}} |\mathrm{FCP}(\mathbf{n})|\mathbf{x}^{\mathbf{n}} = \frac{\prod_{i=1}^{d+1} x_i \left(\sum_{i=1}^{d+1} (x_i^{-1} + d x_i) - \sum_{1\le i,j\le d+1} x_i x_j^{-1} \right)}{\left(1 - \sum_{i=1}^{d+1} x_i\right) \prod_{i=1}^{d+1} (1 - x_i)}. \qquad (1.4)$$

Connections in some other areas

Algebra, geometry, physics

Commutative algebra, Artinian monomial ideals

Enumerative geometry, Donaldson-Thomas invariants (Euler characteristics of Hilbert schemes)

Counting black holes in string theory [Gopakumar-Vafa]

That's it?