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Schur operators and identities for skew stable Grothendieck polynomials

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Abstract. Symmetric Grothendieck polynomials are analogues of Schur polynomials in the K-theory of Grassmannians. We build dual families of symmetric Grothendieck polynomials using Schur operators. This approach allows us to prove the skew Cauchy identity which is our central result. We then derive various consequences and applications: skew Pieri rules, generating series identities, dual filtrations of Young's lattice, and enumerative properties.

Keywords: Grothendieck polynomials, Cauchy identities, dual filtered graphs

1 Introduction

Symmetric Grothendieck polynomials, also known as stable Grothendieck polynomials, are certain K-theoretic deformations of Schur functions. The symmetric Grothendieck polynomial G_λ can be defined as a generating series

$$G_\lambda(x_1, x_2, \ldots) = \sum_T (-1)^{|T|-|\lambda|} \prod_{i \geq 1} x_i^{\#i\text{'s in } T},$$

where the sum runs over *set-valued tableaux*, a generalization of semistandard Young tableaux where boxes may contain sets of integers [4]. These functions were first studied by Fomin and Kirillov [7] as a stable limit of more general Grothendieck polynomials that generalize Schubert polynomials in the other direction.

Being a generalization of the Schur basis, symmetric Grothendieck polynomials share with it many similar properties. However $\{G_\lambda\}$ is *inhomogeneous* and has an unbounded degree when defined for infinitely many variables $(x_1, x_2, \ldots)$, i.e., it lives in the completion $\hat{\Lambda}$ of the ring of symmetric functions Λ. It is thus remarkable that $\{G_\lambda\}$ is a distinguished *basis* for a certain commutative ring which is related to the K-theory of Grassmannians. This was proved by Buch [4] and he gave an analogue of a Littlewood–Richardson rule using set-valued tableaux. Earlier Lenart [9] proved analogues of Pieri rules. Lam and Pylyavskyy [8] gave an explicit formula of the basis $\{g_\lambda\}$ dual to $\{G_\lambda\}$ using plane partitions.

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We study *skew* stable Grothendieck polynomials via noncommutative Schur operators. We used these operators in [16] to prove dualities for certain two-parameter deformations of Grothendieck polynomials. Employing classical Schur operators turns out to be beneficial for obtaining a number of new properties.

We prove the following *skew Cauchy identity* which becomes our central object.

Theorem 1.1. *Let μ, ν be any fixed partitions, then*

$$\sum_{\lambda} G_{\lambda/\!/\mu}(x_1, x_2, \ldots) g_{\lambda/\nu}(y_1, y_2, \ldots) = \prod_{i,j} \frac{1}{1 - x_i y_j} \sum_{\kappa} G_{\nu/\!/\kappa}(x_1, x_2, \ldots) g_{\mu/\kappa}(y_1, y_2, \ldots).$$

For Schur functions such an identity was first given by Zelevinsky in the Russian translation of Macdonald's book. Macdonald [10, Ch. 1 notes] mentioned that this result has apparently been discovered independently by Lascoux, Towber, Stanley, Zelevinsky. It is also known for analogues, e.g., shifted Schur functions [6, 12] (also [14, Ch.7 problems]). Our approach is similar to Fomin's in [6]. Borodin's symmetric functions [2] generalizing Hall-Littlewood polynomials, also satisfy Cauchy identities which is important in certain stochastic models [3]; special cases of these symmetric functions have similarity with Grothendieck polynomials [2].

Using the skew Cauchy identity in both operator and generating function forms, we obtain various nontrivial consequences and applications. We prove *skew Pieri rules* (Theorem 6.1) that in a simplest case give the following formulas

$$g_{(1)} g_{\mu/\nu} = (-i(\mu) + i(\nu)) g_{\mu/\nu} + \sum_{\lambda = \mu + \square} g_{\lambda/\nu} - \sum_{\eta = \nu - \square} g_{\lambda/\eta}$$
$$G_{(1)} G_{\mu/\!/\nu} = \sum_{\substack{\lambda/\mu \text{ rook strip} \\ \eta \subset \nu}} (-1)^{|\lambda/\mu|} G_{\lambda/\!/\eta}.$$

For Schur functions the skew Pieri formula is due to Assaf and McNamara [1]. It is a general principle that skew Cauchy identities allow to obtain skew Pieri formulas which in the case of Hall-Littlewood polynomials was described in [15].

While the homogeneous Schur case corresponds to a graded ring and a combinatorial object behind this is *self-dual graded Young's lattice* [5, 13], Grothendieck polynomials correspond to a filtered ring and *dual filtered graphs* introduced by Patrias and Pylyavskyy in [11]. Our approach gives new types of dual filtered Young graphs. Apparently the most important filtration of dual graded graphs is the so-called *Möbius deformation* [11] as it corresponds to K-theoretic insertion rules. Even though this deformation (sometimes) produces a dual filtered graph, it is unclear why. We show that Möbius deformation of Young's lattice can be obtained from our *Cauchy deformation* by a natural transform related to Möbius inversion. This reveals the presence of a Möbius deformation on Young's lattice. In addition, these constructions give various enumerative identities.

The full version containing detailed proofs and more results accompanying this abstract is in [17].

2 Partitions and Young diagrams

A *partition* is a sequence $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots)$ of nonnegative integers with only finitely many nonzero terms. The weight of a partition λ is the sum $|\lambda| = \lambda_1 + \lambda_2 + \cdots$. Any partition λ is represented as a *Young diagram* which contains λ_i boxes in its ith row $(i = 1, 2, \ldots)$; equivalently it is the set $\{(i, j) : 1 \leq i \leq \ell, 1 \leq j \leq \lambda_i\}$, where $\ell = \ell(\lambda)$ is the number of nonzero parts of λ. We use English notation for Young diagrams, index columns from left to right and rows from top to bottom.

The following terminology and notation will be used throughout. Let $I(\mu)$ be the set of removable boxes of μ. For partitions $\lambda \supset \mu$, let $\lambda /\!/ \mu := \lambda/\mu \cup I(\mu)$. Let $i(\mu) = \#I(\mu)$ be the number of removable boxes of μ (which is the number of *inner corners* of μ).

Let $a(\lambda /\!/ \mu)$ be the number of columns of $\lambda /\!/ \mu$ that are not columns of λ/μ. For example, $a((5331) /\!/ (432)) = 2$. Denote by $c(\lambda/\mu)$ and $r(\lambda/\mu)$ the number of columns and rows of λ/μ, e.g., $c((5331)/(432)) = 3$ and $r((5331)/(433)) = 2$. We also use standard terminology: λ/μ is called a *horizontal (resp. vertical) strip* if no two boxes of λ/μ lie on the same column (resp. row); λ/μ is a *rook strip* if no two boxes lie on the same row or column.

3 Schur operators

Consider the free $\mathbb{Z}$-module $\mathbb{Z}P = \bigoplus_\lambda \mathbb{Z} \cdot \lambda$ with a basis of all partitions.

Definition 3.1. Let $u = (u_1, u_2, \ldots)$ and $d = (d_1, d_2, \ldots)$ be sets of linear operators on $\mathbb{Z}P$, called *Schur operators*, that act on bases for each $i \geq 1$ as follows:

$$u_i \cdot \lambda = \begin{cases} \lambda + \square \text{ on column } i, & \text{if possible,} \\ 0, & \text{otherwise} \end{cases} \qquad d_i \cdot \lambda = \begin{cases} \lambda - \square \text{ on column } i, & \text{if possible,} \\ 0, & \text{otherwise.} \end{cases}$$

The operators u, d build Young diagrams by adding or removing boxes. These operators are noncommutative but they satisfy the following commutation relations that can easily be checked on bases. Let $[a, b] = ab - ba$ denotes the commutator.

Lemma 3.2 ([6]). *The following commutation relations hold for the operators u, d:*

$$\textit{non-local:} \quad [u_j, u_i] = [d_j, d_i] = 0, \ |i - j| \geq 2$$

$$\textit{local Knuth:} \quad [u_{i+1}u_i, u_i] = [u_{i+1}u_i, u_{i+1}] = [d_i d_{i+1}, d_i] = [d_i d_{i+1}, d_{i+1}] = 0, (i \geq 1)$$

$$\textit{conjugate:} \quad [d_j, u_i] = 0 \ (i \neq j), \ d_{i+1}u_{i+1} = u_i d_i \ (i \geq 1), \ d_1 u_1 = 1.$$

Schur operators build Schur polynomials and provide a unified approach for studying their various properties such as Cauchy identities and RSK [6]. We generalize this approach for a K-theoretic setting of Grothendieck polynomials.

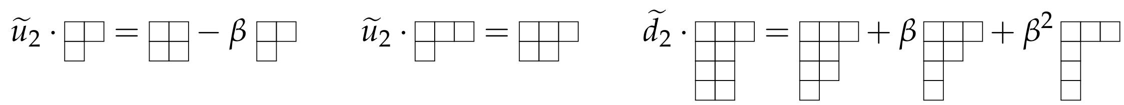
3.1 Grothendieck-Schur operators

Let β be a (scalar) parameter and consider the free $\mathbb{Z}[\beta]$-module $\mathbb{Z}[\beta]P = \bigoplus_\lambda \mathbb{Z}[\beta] \cdot \lambda$.

Definition 3.3. Let $\widetilde{u} = (\widetilde{u}_1, \widetilde{u}_2, \ldots)$ and $\widetilde{d} = (\widetilde{d}_1, \widetilde{d}_2, \ldots)$ be linear operators acting on $\mathbb{Z}[\beta]P$ and defined via the Schur operators as follows:

$$\widetilde{u}_i := u_i - \beta u_i d_i = u_i(1 - \beta d_i), \qquad \widetilde{d}_i := d_i + \beta d_i^2 + \beta^2 d_i^3 + \cdots = (1 - \beta d_i)^{-1} d_i.$$

For example,



For a diagram, the operator $\widetilde{u}_i$ adds a box on ith column if possible *or* applies the following *loop* condition: if the lowest box in the ith column is removable it multiplies the result by $-\beta$, since the operator $u_i d_i$ results 1 (an identity) if the box in the ith column is removable, and 0 otherwise. The operator $\widetilde{d}_i$ removes boxes from the ith column graded by β in all possible ways. We defined these operators in [16] when we studied duality properties of stable Grothendieck polynomials.

Lemma 3.4. *The following commutation relations hold for the operators $\widetilde{u}, \widetilde{d}$:*

non-local: $[\widetilde{u}_i, \widetilde{u}_j] = [\widetilde{d}_i, \widetilde{d}_j] = 0, \ |i - j| \geq 2$

local: $[\widetilde{u}_{i+1}\widetilde{u}_i, \widetilde{u}_i + \widetilde{u}_{i+1}] = [\widetilde{d}_i\widetilde{d}_{i+1}, \widetilde{d}_i + \widetilde{d}_{i+1}] = 0 \ (i \geq 1)$

conjugate: $[\widetilde{u}_i, \widetilde{d}_j] = 0 \ |i - j| \geq 2, \ [\widetilde{u}_{i+1}, \widetilde{d}_i] = 0 \ (i \geq 1), \ \widetilde{d}_1\widetilde{u}_1 = 1.$

Remark 3.5. For $\beta = 0$ everything turns into Schur operators. In general, relations for $\widetilde{u}, \widetilde{d}$ are more complicated than for u, d. E.g., the proof of local relations uses the identity $[u_i d_i, u_{i+1} u_i] = 0$ for Schur operators that is not from the list of Lemma 3.2.

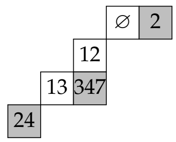
4 Symmetric skew Grothendieck polynomials

Let x be an indeterminate (central variable, commuting with the u, d) and define the series

$$A(x) = \cdots (1 + x\widetilde{u}_2)(1 + x\widetilde{u}_1), \qquad B(x) = (1 + x\widetilde{d}_1)(1 + x\widetilde{d}_2) \cdots$$

From non-local and local relations given in Lemma 3.4 it is standard (e.g., [6, 7]) to deduce that (for commuting x, y)

$$[A(x), A(y)] = 0, \quad [B(x), B(y)] = 0.$$



$$x^T = x_1^2 x_2^3 x_3^2 x_4^2 x_7 \qquad |T| = 10$$

Figure 1: A set-valued tableau of shape $(5331)//(432) = (5331)/(432) \cup I(432)$. The gray boxes form the skew shape $(5331)/(432)$; the white cells are the removable boxes $I(432)$ of (432). It is allowed to put $\varnothing$ in $I(\mu)$ but boxes of λ/μ must be nonempty.

Definition 4.1. Define the *skew Grothendieck polynomials* $\{G^{\beta}_{\lambda//\mu}\}, \{g^{\beta}_{\lambda/\mu}\}$ via the series

$$A(x_n)\cdots A(x_1)\mu = \sum_{\lambda} G^{\beta}_{\lambda//\mu}(x_1,\ldots,x_n)\cdot\lambda, \quad B(x_n)\cdots B(x_1)\lambda = \sum_{\mu} g^{\beta}_{\lambda/\mu}(x_1,\ldots,x_n)\cdot\mu.$$

Since the series $[A(x_i), A(x_j)] = [B(x_i), B(x_j)] = 0$ commute, the functions G^{β}, g^{β} (indexed by pairs of partitions) are well-defined polynomials *symmetric* in $x_1,\ldots,x_n$. We can then extend these symmetric functions for infinitely many variables $(x_1, x_2, \ldots)$ by letting $n \to \infty$. Notice that $G^{\beta}_{\lambda//\mu} = g^{\beta}_{\lambda/\mu} = 0$ if $\mu \not\subseteq \lambda$.

Remark 4.2. The reason why we use the notation $G_{\lambda//\mu}$ and *not* $G_{\lambda/\mu}$ is in boundary conditions, e.g., $G_{\lambda//\lambda} = \prod_k (1-x_k)^{i(\lambda)} \neq G_{\varnothing} = 1$. We are also consistent with Buch's notation [4] (though he defines these functions in a different way).

Proposition 4.3. *The following branching formulas hold:*

$$G^{\beta}_{\lambda//\mu}(x_1,\ldots,x_n,y_1,\ldots,y_m) = \sum_{\nu} G^{\beta}_{\lambda//\nu}(x_1,\ldots,x_n) G^{\beta}_{\nu//\mu}(y_1,\ldots,y_m),$$
$$g^{\beta}_{\lambda/\mu}(x_1,\ldots,x_n,y_1,\ldots,y_m) = \sum_{\nu} g^{\beta}_{\lambda/\nu}(x_1,\ldots,x_n) g^{\beta}_{\nu/\mu}(y_1,\ldots,y_m).$$

A *set-valued tableaux* (SVT) of shape $\lambda//\mu$ is a filling of a shape $\lambda//\mu = \lambda/\mu \cup I(\mu)$ (i.e., skew shape λ/μ and removable boxes of μ) by sets of positive integers such that if one replaces each set by any of its elements the resulting tableau is *semistandard* (i.e., has weakly increasing rows from left to right and strictly increasing columns from top to bottom). When filling λ/μ, sets in boxes should be nonempty, however when filling the boxes of $I(\mu)$ it is allowed to have empty sets. For a set-valued tableau T, the corresponding monomial is defined as $x^T = \prod_{i\geq 1} x_i^{a_i}$, where a_i is the number of i's in T and let $|T| = \sum_i a_i$. See Fig. 1.

A *reverse plane partition* (RPP) of shape λ/μ is a filling of a Young diagram of λ/μ by positive integers weakly increasing in rows from left to right and columns from top to bottom. For a reverse plane partition T, the corresponding monomial is defined as $x^T = \prod_{i\geq 1} x_i^{c_i}$ where c_i is the number of columns of T containing i and let $|T| = \sum_i c_i$.

Theorem 4.4. *The following combinatorial formulas hold:*

$$G^{\beta}_{\lambda/\!/\mu} = \sum_{T \in SVT(\lambda/\!/\mu)} (-\beta)^{|T|-|\lambda/\mu|} x^T, \qquad g^{\beta}_{\lambda/\mu} = \sum_{T \in RPP(\lambda/\mu)} \beta^{|\lambda/\mu|-|T|} x^T.$$

Remark 4.5. Using the the operators $\widetilde{u}, \widetilde{d}$ we obtained the formulas due to Buch [4] and Lam-Pylyavskyy [8]. For $\mu = \varnothing$ and $\beta = 1$ they coincide with dual families of stable Grothendieck polynomials, $\{G_\lambda\}, \{g_\lambda\}$. They are Hopf-dual or dual via the Hall inner product $\langle G_\lambda, g_\mu \rangle = \delta_{\lambda\mu}$ for which the Schur functions form an orthonormal basis.

Remark 4.6. Let $\beta = 1$ and $G_{\lambda/\mu}$ be the symmetric function defined via SVT formula of skew shape λ/μ (without removable boxes). Then the two functions are related [4] via the Möbius inversion

$$G_{\lambda/\mu} = \sum_{\nu \subset \mu} G_{\lambda/\!/\nu}, \qquad G_{\lambda/\!/\nu} = \sum_{\nu/\mu \text{ rook strip}} (-1)^{|\nu/\mu|} G_{\lambda/\mu}.$$

5 Skew Cauchy identity

Theorem 5.1. *Let μ, ν be any fixed partitions, then*

$$\sum_{\lambda} G^{\beta}_{\lambda/\!/\mu}(\mathbf{x}) g^{\beta}_{\lambda/\nu}(\mathbf{y}) = \prod_{i,j} \frac{1}{1 - x_i y_j} \sum_{\kappa} G^{\beta}_{\nu/\!/\kappa}(\mathbf{x}) g^{\beta}_{\mu/\kappa}(\mathbf{y}). \tag{5.1}$$

This identity is equivalent to the following commutation relation for the series A, B.

Theorem 5.2. *The following commutation relation holds*

$$B(y)A(x) = \frac{1}{1 - xy} A(x)B(y).$$

The proof of Theorem 5.2 uses the following Yang-Baxter-type local identity for the operators $\widetilde{u}, \widetilde{d}$.

Lemma 5.3. *For all $i \geq 1$ we have*

$$(1 - xy\widetilde{u}_i\widetilde{d}_i)^{-1}(1 + x\widetilde{u}_i)(1 + y\widetilde{d}_{i+1}) = (1 - xy\widetilde{d}_{i+1}\widetilde{u}_{i+1})^{-1}(1 + y\widetilde{d}_{i+1})(1 + x\widetilde{u}_i).$$

5.1 Corollaries

First note that by setting $\beta = 0$, the results generalize corresponding properties of skew Schur polynomials, since $G^0_{\lambda/\!/\mu} = g^0_{\lambda/\mu} = s_{\lambda/\mu}$.

Corollary 5.4 (Cauchy identity)**.** For $\mu = \nu = \varnothing$ we obtain the usual Cauchy identity

$$\sum_{\lambda} G^{\beta}_{\lambda}(\mathbf{x}) g^{\beta}_{\lambda}(\mathbf{y}) = \prod_{i,j} \frac{1}{1 - x_i y_j}.$$

This identity is equivalent to the duality of the families $\{G_\lambda\}, \{g_\mu\}$ via the standard Hall inner product, i.e., $\langle G_\lambda, g_\mu \rangle = \delta_{\lambda\mu}$.

Corollary 5.5 (Pieri-type formulas)**.** For $\mu = \varnothing$ or $\nu = \varnothing$ we obtain

$$\sum_{\lambda} G^{\beta}_{\lambda}(\mathbf{x}) g^{\beta}_{\lambda/\nu}(\mathbf{y}) = \prod_{i,j} \frac{1}{1 - x_i y_j} G^{\beta}_{\nu}(\mathbf{x}), \qquad \sum_{\lambda} G^{\beta}_{\lambda/\!/\mu}(\mathbf{x}) g^{\beta}_{\lambda}(\mathbf{y}) = \prod_{i,j} \frac{1}{1 - x_i y_j} g^{\beta}_{\mu}(\mathbf{y}).$$

In particular, the last identity gives

$$\sum_{\lambda} \beta^{|\lambda| - c(\lambda)} G^{\beta}_{\lambda/\!/\mu} = \beta^{|\mu| - c(\mu)} \prod_{i} \frac{1}{1 - x_i}, \qquad \sum_{\lambda} G_{\lambda/\!/\mu} = \prod_{i} \frac{1}{1 - x_i}.$$

Recall in contrast a similar identity for Schur functions (e.g., [10]):

$$\sum_{\lambda} s_{\lambda/\mu} = \prod_{i} \frac{1}{1 - x_i} \prod_{i<j} \frac{1}{1 - x_i x_j} \sum_{\kappa} s_{\mu/\kappa}.$$

Let $d(\lambda) := \#\{\mu : \mu \subset \lambda\}$ be the number of subdiagrams of λ. For a 'pure' skew shape λ/μ we have the following generating series.

Theorem 5.6. *We have*

$$\sum_{\lambda} G_{\lambda/\mu} = d(\mu) \prod_{i} \frac{1}{1 - x_i}, \qquad \sum_{\lambda} d(\lambda) G_{\lambda} = \prod_{i} \frac{1}{(1 - x_i)^2}$$

In particular we obtain a curious identity involving the *Catalan numbers* Cat_n.

$$\sum_{\lambda} G_{\lambda/\delta_n} = \mathrm{Cat}_n \prod_{i} \frac{1}{1 - x_i}, \qquad \delta_n = (n, n-1, \ldots, 1).$$

6 Skew Pieri formulas

The skew Cauchy identity can be used to obtain skew Pieri formulas.

Theorem 6.1 (Skew Pieri rules)**.** *We have*

$$G^{\beta}_{(1^k)} G^{\beta}_{\mu/\!/\nu} = \sum_{\substack{\lambda/\mu \text{ vert strip} \\ \eta \subset \nu}} W^{\lambda,\mu}_{\nu,\eta}\, G^{\beta}_{\lambda/\!/\eta} \quad W^{\lambda,\mu}_{\nu,\eta} = (-1)^{|\lambda/\mu| - k} \beta^{|\lambda/\mu| + |\nu/\eta| - k} \binom{c(\lambda/\mu) + c(\nu/\eta) - 1}{|\lambda/\mu| + c(\nu/\eta) - k}$$

$$g^{\beta}_{(k)} g^{\beta}_{\mu/\nu} = \sum_{\substack{\lambda/\mu \text{ hor strip} \\ \nu/\eta \text{ vert strip}}} w^{\lambda,\mu}_{\nu,\eta}\, g^{\beta}_{\lambda/\eta} \qquad w^{\lambda,\mu}_{\nu,\eta} = (-1)^{k - |\lambda/\mu|} \beta^{k - |\lambda/\mu| - |\nu/\eta|} \binom{a(\lambda/\!/\mu) - a(\nu'/\!/\eta') - \nu/\eta}{k - |\lambda/\mu| - |\nu/\eta|}$$

Note that these expansions are finite. One could interpret the coefficients w, W as a number of walks in certain dual graphs given in the next section. Notice also that by applying the automorphisms $\tau : g_\lambda \mapsto g_{\lambda'}$, $\widehat{\tau} : G_\lambda \mapsto G_{\lambda'}$, one can show that $\tau(g_{\lambda/\mu}) = g_{\lambda'/\mu'}$, $\widehat{\tau}(G_{\lambda/\!/\mu}) = G_{\lambda'/\!/\mu'}$ and hence the formulas give dual skew Pieri rules as well.

Corollary 6.2 (Simple skew Pieri rules)**.** Let $\beta = 1$. We have

$$g_{(1)} g_{\mu/\nu} = (-i(\mu) + i(\nu)) g_{\mu/\nu} + \sum_{\lambda = \mu + \square} g_{\lambda/\nu} - \sum_{\eta = \nu - \square} g_{\lambda/\eta}$$

$$G_{(1)} G_{\mu/\!/\nu} = \sum_{\substack{\lambda/\mu \text{ rook strip} \\ \eta \subset \nu}} (-1)^{|\lambda/\mu|} G_{\lambda/\!/\eta}.$$

Corollary 6.3. $\beta = 0$ gives the skew Pieri rule for $s_{\lambda/\mu}$ [1]

$$s_{(k)} s_{\mu/\nu} = \sum_{\substack{\lambda/\mu \text{ hor strip} \\ \nu/\eta \text{ vert strip}}} (-1)^{|\nu/\eta|} s_{\lambda/\eta}, \qquad |\lambda/\mu| + |\nu/\eta| = k.$$

Let $J_{\lambda/\!/\mu} := \omega(G_{\lambda/\!/\mu})$ and $j_{\lambda/\mu} := \omega(g_{\lambda/\mu})$, where ω is the standard involution defined on the Schur basis by $\omega : s_\lambda \mapsto s_{\lambda'}$ (in case of $G \in \hat{\Lambda}$ it is extended in the same manner but for infinite linear combinations).

Theorem 6.4 (Skew Pieri-type formulas)**.** *We have*

$$\sum_{\lambda,\eta} j_{\nu/\eta}(-\mathbf{y}) g_{\lambda/\mu}(\mathbf{y}) G_{\lambda/\!/\eta}(\mathbf{x}) = \prod_{i,j} \frac{1}{(1 - x_i y_j)} G_{\mu/\!/\nu}(\mathbf{x}),$$

$$\sum_{\lambda,\eta} J_{\nu/\!/\eta}(-\mathbf{x}) G_{\lambda/\!/\mu}(\mathbf{x}) g_{\lambda/\eta}(\mathbf{y}) = \prod_{i,j} \frac{1}{(1 - x_i y_j)} g_{\mu/\nu}(\mathbf{y}).$$

7 Dual filtered Young graphs

Following [11], a *weighted filtered graph* is a digraph $G = (V, r, E, w)$ where V is a set of countably many vertices together with a *rank function* $r : V \to \mathbb{Z}$ satisfying $r(a) \le r(b)$ for every (directed) edge $(a, b) \in E$, and $w : E \to \mathbb{R}$ is some weight function.

For a pair $G_1 = (V, r, E_1, w_1), G_2 = (V, r, E_2, w_2)$ of filtered graphs on the same (ranked) vertex set V construct a digraph $G = (V, E)$ so that $E = E_1 \cup \overline{E}_2$ is a union of edges E_1 and edges of E_2 but taken in *opposite* direction. Let $\mathbb{R}V$ be the free abelian group on V (formal $\mathbb{R}$-linear combinations of vertices V). Define the *up* and *down* operators $U, D \in End(\mathbb{R}V)$ on G as follows

$$Uv = \sum_{e = (v \to u) \in E} w_1(e) u, \qquad Dv = \sum_{e = (u \to v) \in E} w_2(e) u.$$

Say that G is a *dual filtered graph* if there exist $\alpha, \beta \in \mathbb{R}$ such that for every $v \in V$ we have

$$[D, U]v = (DU - UD)v = (\alpha + \beta D)v.$$

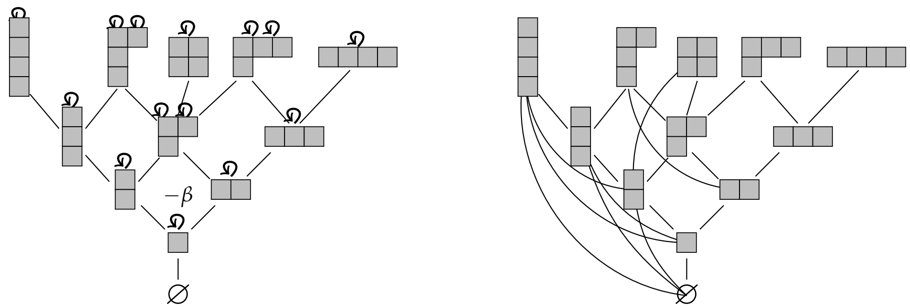


Figure 2: Dual filtered Young graph $\widetilde{\mathbb{Y}}$. The graph on the left corresponds to *up* edges and on the right to *down* edges. Here each loop has weight $-\beta$.

Remark 7.1. By normalizing the operators one can see that there are only three distinct types of (α, β) that are $(1,1),(0,1),(1,0)$. In addition, for the relation $[D, U] = D$, shifting $D' = D - 1$ gives $[D', U] = 1 + D'$.

Remark 7.2. If the rank function r satisfies $r(a) + 1 = r(b)$ for every edge (a, b) and $(\alpha, \beta) = (1, 0)$, then the corresponding graphs G are called *dual graded graphs* studied by Fomin [5] and by Stanley [13] as *differential posets*.

Remark 7.3. An associative algebra generated by U, D subject to $[D, U] = 1$ is called the *first Weyl algebra*. We may consider $D = \frac{d}{dx}$ as a differential operator and $U = x$ acting on a polynomial ring $K[x]$. The relation $[D, U] = 1 + D$ corresponds to the difference operator $Df(x) = f(x+1) - f(x)$.

7.1 New constructions of filtered Young graphs

Let $\mathbb{Y}$ be the Young lattice, i.e., an infinite graph whose vertices are indexed by partitions and edges are given by $(\lambda, \lambda + \square)$. We think of $\mathbb{Y}$ as a self-dual graph with up and down directed edges $(\lambda \to \lambda \pm \square)$.

First define the following *β-filtration* $\widetilde{\mathbb{Y}}$ of Young's lattice $\mathbb{Y}$ (see Fig. 2):

(i) vertices V are integer partitions ranked by the number of boxes $r(\lambda) = |\lambda|$

(ii) *up* edges (of E_1) are as in Young's lattice $(\lambda \to \lambda + \square)$ with the weight $w = 1$ but there are also $i(\lambda)$ many *loops* $(\lambda \to \lambda)$ each with the weight $w = -\beta$ (recall that $i(\lambda)$ is the number of inner corners of λ)

(iii) *down* edges (of $\overline{E}_2$) are given by $(\lambda \to \mu)$ iff all boxes λ/μ are on a single column, and the corresponding weight is $w = \beta^{|\lambda/\mu|-1}$.

Next, the *Cauchy filtration* of $\mathbb{Y}$ denoted by $\overline{\mathbb{Y}}$ satisfies the same conditions (i) and (ii) as for $\widetilde{\mathbb{Y}}$ but its *down* edges are given by $(\lambda \to \mu)$ iff $\lambda \supset \mu$ and the weight is $w = \beta^{|\lambda/\mu| - c(\lambda/\mu)}$.

Theorem 7.4. *We have: (i)* $\widetilde{\mathbb{Y}}$ *is a dual filtered graph satisfying* $[D,U] = 1$*. (ii)* $\overline{\mathbb{Y}}$ *is a dual filtered graph satisfying* $[D,U] = 1 + D$*.*

Corollary 7.5. For $\beta = 0$ we have the following special cases: (i) $\widetilde{\mathbb{Y}} = \mathbb{Y}$ is the self-dual graded Young graph. (ii) $\overline{\mathbb{Y}}$ gives the *Pieri deformation* of Young's graph: up edges are as in the usual Young's graph $\mathbb{Y}$ and down edges are given by $(\lambda \to \mu)$ iff λ/μ is a horizontal strip.

These constructions are natural consequences of Cauchy identity. Suppose the operator series

$$A(x) = 1 + \sum_{i \geq 1} U_i x^i, \qquad B(y) = 1 + \sum_{i \geq 1} D_i y^i,$$

satisfy the Cauchy identity $B(y)A(x) = (1 - xy)^{-1} A(x)B(y)$. Then by comparing the coefficients at xy and x after plugging $y = 1$ we obtain that

$$[D_1, U_1] = 1 \qquad \text{and} \qquad [D_\infty, U_1] = 1 + D_\infty, \quad D_\infty = D_1 + D_2 + \cdots.$$

Let $\beta \in \mathbb{R}$ and define the operators

$$\widetilde{U} = \widetilde{u}_1 + \widetilde{u}_2 + \cdots, \qquad \widetilde{D} = \widetilde{d}_1 + \widetilde{d}_2 + \cdots, \qquad \overline{D} = -1 + (1 + \widetilde{d}_1)(1 + \widetilde{d}_2) \cdots$$

As shown just above, the Cauchy identity gives $[\widetilde{D}, \widetilde{U}] = 1$ and $[\overline{D}, \widetilde{U}] = 1 + \overline{D}$.

It was noted in [11] that apparently the most interesting and mysterious type of dual filtered graphs is the so-called *Möbius deformation* that is defined for Young's lattice as follows. Again, the defining conditions (i) and (ii) are the same as for β-filtration $\widetilde{\mathbb{Y}}$ but loops have *positive* weight 1, and *down* edges are given by $(\lambda \to \mu)$ iff λ/μ is a rook strip (i.e., no two boxes lie on the same row or column) and the corresponding weight $w = 1$.

Besides providing new examples of dual filtered Young's graphs, a consequence of our approach is the following result revealing the presence of the Möbius deformation of Young's lattice: it is related to the Cauchy deformation and can be obtained from it via a natural transformation.

Theorem 7.6. $\widehat{\mathbb{Y}}$ *is a dual filtered graph satisfying* $[D,U] = (1 + D)$*. Moreover,* $\widehat{\mathbb{Y}}$ *can be obtained from the Cauchy deformation* $\overline{\mathbb{Y}}$ *with down edges weighted by* $(-1)^{|\lambda/\mu|}$ *via the map*

$$D \longmapsto D(1 + D)^{-1}.$$

8 Enumeration formulas

Define *increasing set-valued tableaux* (ISVT) as an SVT that if after replacing each set by any of its element, the resulting tableau is increasing both in rows and columns. Let $F_{\lambda /\!/ \mu}(n)$ be the number of ISVT of shape $\lambda /\!/ \mu$ that contains all numbers from $[n] := \{1, \ldots, n\}$.

A *strict tableaux* (ST) of skew shape λ/μ is a filling of a Young diagram of λ/μ by positive integers so that entries strictly increase in rows from left to right, weakly increase from top to bottom, and each element can appear only on a single column. Let $f_{\lambda/\mu}(n)$ be the number of ST of shape λ/μ that contain all numbers from $[n]$.

An *increasing tableaux* (IT) is a filling of a skew diagram by positive integers so that they strictly increase in both rows and columns. Let now $g_{\lambda/\mu}(n)$ be the number of IT of skew shape λ/μ that contain all numbers $[n]$ (some numbers may appear several times).

Theorem 8.1. *We have*

$$\sum_{\lambda}(-1)^{n-|\lambda/\mu|}F_{\lambda/\!/\mu}(n)f_{\lambda/\nu}(m) = \sum_i i!\binom{m}{i}\binom{n}{i}\sum_{\kappa}(-1)^{m-i-|\nu/\kappa|}F_{\nu/\!/\kappa}(n-i)f_{\mu/\kappa}(m-i),$$

$$\sum_{\lambda}F_{\lambda/\!/\mu}(n)g_{\lambda/\nu}(m) = \sum_{i,j}q_n(i,j)\binom{m}{i}\binom{n}{j}\sum_{\kappa}F_{\nu/\!/\kappa}(n-j)g_{\mu/\kappa}(m-i),$$

where

$$q_n(i,j) = \sum_{\ell}\binom{i-j+\ell}{\ell}A_{i,n-\ell}$$

and $A_{i,s}$ is the Eulerian number, i.e., the number of permutations of $(1,\ldots,i)$ with s descents.

These formulas are applications of the normal ordering of differential operators U, D and analogous to the formula (see [12, 13])

$$\sum_{\substack{|\lambda/\mu|=n\\|\lambda/\nu|=m}} f_{\lambda/\mu}f_{\lambda/\nu} = \sum_{i\geq 0} i!\binom{m}{i}\binom{n}{i}\sum_{\substack{|\nu/\kappa|=n-i\\|\mu/\kappa|=m-i}} f_{\nu/\kappa}f_{\mu/\kappa},$$

where $f_{\lambda/\mu}$ is the number of standard Young tableaux (SYT) of shape λ/μ. It generalizes the classical identity

$$\sum_{\lambda\vdash n} f_{\lambda}^2 = n!$$

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