DETERMINANTAL FORMULAS FOR DUAL GROTHENDIECK POLYNOMIALS

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ABSTRACT. We prove Jacobi-Trudi-type determinantal formulas for skew dual Grothendieck polynomials which are K-theoretic deformations of Schur polynomials. We also prove a bialternant type formula analogous to the classical definition of Schur polynomials.

1. INTRODUCTION

The *dual* (stable) *Grothendieck polynomials* are certain K-theoretic inhomogeneous deformations of Schur polynomials, introduced by Lam and Pylyavskyy in [LP07] (with earlier implicit relations in [Len00, Buc02], as a dual basis to stable Grothendieck polynomials [FK94]). Besides interesting combinatorial properties [LP07, GGL16, Yel17, Yel19, Yel19b], these functions have connections with some natural probabilistic models [Yel19a, Yel20, Yel20a], and hence they arise in contexts beyond K-theoretic Schubert calculus.

While various aspects of these polynomials are developed well (e.g. in the references above), their theory can still be regarded as incomplete.[1] One gap concerned problems about determinantal formulas (for skew shapes) which, of course, have fundamental importance.

In this paper, we prove several determinantal identities for the dual Grothendieck polynomials $g_{\lambda/\mu}$. Let us state one of our main results.

Theorem 1 (Jacobi-Trudi-type formula). *Let $n \geq \lambda_1, \mu_1$. The following determinantal identity holds* [2]

(1)
$$g_{\lambda/\mu}(\mathbf{x}) = \det \left[e_{\lambda'_i - i - \mu'_j + j}(1^{\lambda'_i - 1 - \mu'_j}, \mathbf{x}) \right]_{1 \leq i,j \leq n}.$$

This formula is of Jacobi-Trudi type or an analogue of the Nägelsbach-Kostka formula[3] for Schur functions, see e.g. [Mac98, Sta99]. In the straight shape case $\mu = \varnothing$, it was proved by the second author in [Yel17]. In a more general refined form which we also prove in Sec. 3, this identity was conjectured by Grinberg [Gri15]. In fact, Jacobi-Trudi-type formulas for g_λ were first presented in [SZ03].

We also prove a dual formula (Theorem 16) written via the complete homogeneous symmetric functions $\{h_n\}$, and derive a bialternant-type formula (Theorem 19) which is analogous to the classical definition of Schur polynomials.

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[1] In contrast with the plethora of results known for Schur functions. See also the final section with some open questions.

[2] Here $\mathbf{x} = (x_1, x_2, \ldots)$; e_n is the elementary symmetric function; λ' is the conjugate partition of λ; and $1^k := (1, \ldots, 1)$ repeated k times which is defined as $\varnothing$ for $k \leq 0$.

[3] Which is usually referred to as the dual Jacobi-Trudi identity.

The proof of Theorem 1 is quite involved.[4] Combinatorial proofs of such identities always rely on the Lindström-Gessel-Viennot (LGV) lemma [GV89] via lattice paths. But perhaps the most surprising aspect of our proof is a 3-dimensional lattice construction. This system needs modification of the LGV method using certain path transformations and one of the most difficult parts of the proof is in describing the sign-reversing involution. Even though the operations look somewhat complicated, they are essentially like *jeu de taquin* type operations on tableaux. See Sec. 3 for details. We also note that a different and independent proof of this result was posted by J. S. Kim [Kim20] almost the same time as our paper was written.

2. Preliminaries

2.1. **Partitions and diagrams.** A *partition* is a sequence $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ of positive integers $\lambda_1 \geq \cdots \geq \lambda_\ell$, where $\ell(\lambda) = \ell$ is the *length* of λ (we set $\lambda_i = 0$ for $i > \ell$). Every partition λ can be represented as the *Young diagram* $\{(i,j) : i \in [1,\ell], j \in [1,\lambda_i]\}$. We denote by λ' the *conjugate* partition of λ whose diagram is transposed. We draw diagrams in English notation. For $\lambda \supset \mu$, the *skew shape* λ/μ has the diagram of μ removed from the diagram of λ.

2.2. **Dual Grothendieck polynomials.** A *reverse plane partition* (RPP) of shape λ/μ is a filling of the boxes of the diagram of λ/μ with positive integers weakly increasing along rows from left to right and along columns from top to bottom. Let $\mathrm{RPP}(\lambda/\mu)$ be the set of RPP's of the skew shape λ/μ.

Definition 2 ([LP07])**.** The *skew dual Grothendieck polynomials* $g_{\lambda/\mu}$ are defined as follows:

$$g_{\lambda/\mu}(\mathbf{x}) := \sum_{T \in \mathrm{RPP}(\lambda/\mu)} \prod_{i \geq 1} x_i^{c_i(T)},$$

where $c_i(T)$ is the number of columns of T containing the entry i, and $\mathbf{x} = (x_1, x_2, \ldots)$.

It is known that $g_{\lambda/\mu}$ is a symmetric function [LP07, GGL16, Yel19], which is inhomogeneous and its top degree component is the Schur polynomial $s_{\lambda/\mu}$.

3. Proof of Theorem 1

3.1. **Refined version.** We will prove a more general version of Theorem 1 for *refined* dual Grothendieck polynomials $g_{\lambda/\mu}(\mathbf{t}; \mathbf{x})$ introduced in [GGL16] which contain the extra parameters $\mathbf{t} = (t_1, t_2, \ldots)$ so that $g_{\lambda/\mu}(\mathbf{1}; \mathbf{x}) = g_{\lambda/\mu}(\mathbf{x})$ (i.e. when $t_i = 1$ for all i). We define these formal power series as follows

$$g_{\lambda/\mu}(\mathbf{t}; \mathbf{x}) := \sum_{T \in \mathrm{RPP}(\lambda/\mu)} \prod_{i \geq 1} x_i^{c_i(T)} \prod_{j \geq 1} t_j^{d_j(T)},$$

where $d_j(T)$ is the number of entries in j-th row of T that are equal to an entry directly below. The following generalization of Theorem 1 was conjectured by Grinberg [Gri15].

Theorem 3. *Let $n \geq \lambda_1, \mu_1$. The following determinantal identity holds*

(2) $$g_{\lambda/\mu}(\mathbf{t}; \mathbf{x}) = \det \left[e_{\lambda'_i - i - \mu'_j + j}(t_{\mu'_j+1}, \ldots, t_{\lambda'_i-1}, \mathbf{x}) \right]_{1 \leq i,j \leq n}.$$

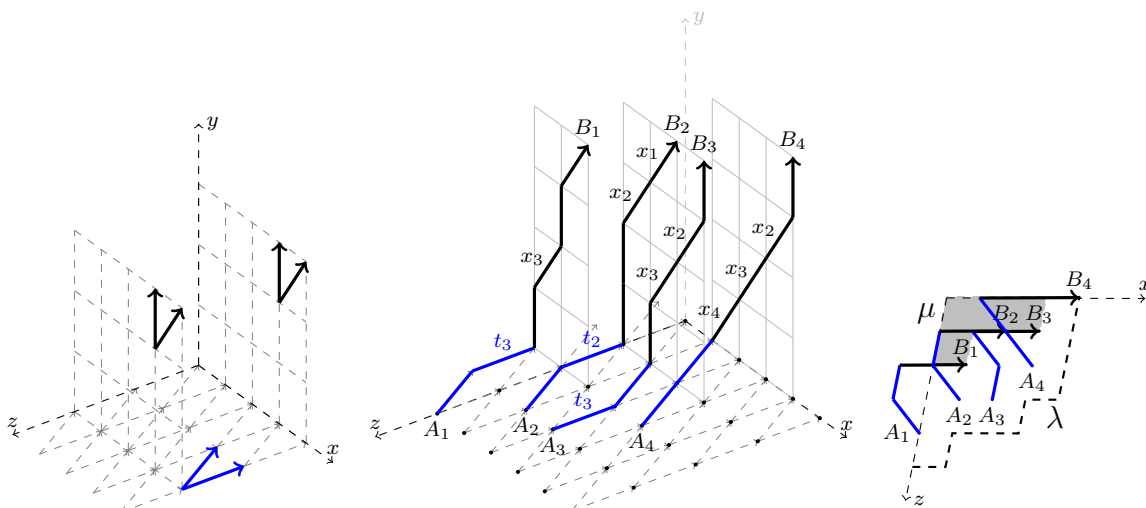[4] It is significantly more difficult than the straight shape case proved in [Yel17].

FIGURE 1. (a) A part of the lattice with typical sample steps. (b) An example of a path system for $\lambda'/\mu' = (5443)/(211)$, $n = 4$ and $m = 4$ with the path weights given by $w(P_1) = t_3 x_1 x_3$, $w(P_2) = t_2 x_1 x_2$, $w(P_3) = t_3 x_2 x_3$, $w(P_4) = x_2 x_3 x_4$. (c) A view of (b) from above with λ and μ shown.

3.2. **Proof overview.** First, we present a 3d lattice path system whose path enumerators give the r.h.s. of (2). Then the most difficult part of the proof contains the description of a sign-reversing weight-preserving involution on such path systems which cancels out negative terms. To describe it, we introduce several operations on path systems (subsec. 3.4). Finally, we show a correspondence between *good* path systems (fixed by the involution) and skew RPPs, which gives the formula for refined dual Grothendieck polynomials.

3.3. **A 3d lattice path system.** Let m be the number of variables in $\mathbf{x}$ (assumed finite for now, we can afterwards let $m \to \infty$). Let λ, μ and $n \geq \lambda_1, \mu_1$ be all fixed. Consider the lattice $\mathbb{Z}^3$ with the following weighted edges e (which we call *steps*):

- on the planes $z = k$ (the *walls*):
  - $(i, j, k) \to (i, j + 1, k)$ with the weight $w(e) = 1$;
  - $(i, j, k) \to (i + 1, j + 1, k)$ with the weight $w(e) = x_{m-j}$ for $j \in [0, m - 1]$;
- on the plane $y = 0$ (the *floor*):
  - $(i, 0, j) \to (i, 0, j - 1)$ with the weight $w(e) = t_j$ for $j > 0$;
  - $(i, 0, j) \to (i - 1, 0, j - 1)$ with the weight $w(e) = 1$.

(All other weights are 0.) See Fig. 1. Note that any (nonzero weight) path which starts on the floor plane and ends on a wall plane, begins with steps on the floor plane (each decrementing the z-coordinate) and then continues with steps on the wall plane (incrementing the y-coordinate).

Let us define n *source* points $\mathbf{A} = (A_1, \ldots, A_n)$ and n *sinks* $\mathbf{B} = (B_1, \ldots, B_n)$ whose coordinates are given by $A_i = (i - 1, 0, \lambda_i' - 1)$ and $B_i = (i, m, \mu_i')$ for all $i \in [1, n]$. Define weighted path enumerators in the usual way

$$w(A \to B) := \sum_{P : A \to B} \prod_{e \in P} w(e)$$

over all paths P in the lattice from the point A to the point B with steps e given as above. The following formula is then clear from the construction.

Lemma 4. *We have*

$$w(A_i \to B_j) = e_{\lambda_i' - i - \mu_j' + j}(t_{\mu_j'+1}, \ldots, t_{\lambda_i'-1}, x_1, \ldots, x_m).$$

Similarly, define the signed weighted multi-enumerators

(3) $$w(\mathbf{A} \to \mathbf{B}) := \sum_{\mathbf{P} \in N(\mathbf{A}, \mathbf{B})} \mathrm{sgn}(\mathbf{P})\, w(\mathbf{P}),$$

where $N(\mathbf{A}, \mathbf{B})$ is the set of nonintersecting path systems $\mathbf{P}$ from $\mathbf{A}$ to $\mathbf{B}$ (i.e. n paths with no common vertices as in Fig. 1(b)), $\mathrm{sgn}(\mathbf{P}) := \mathrm{sgn}(\sigma)$ for $\sigma \in S_n$ if $\mathbf{P}$ joins A_i with $B_{\sigma(i)}$ for all $i \in [1, n]$, and $w(\mathbf{P})$ is the product of weights of all edges in the whole system $\mathbf{P}$.

Since our directed graph is acyclic, by the LGV lemma we immediately obtain the following.

Corollary 5. *We have*

$$\det\left[e_{\lambda_i' - i - \mu_j' + j}(t_{\mu_j'+1}, \ldots, t_{\lambda_i'-1}, x_1, \ldots, x_m)\right]_{1 \le i, j \le n} = w(\mathbf{A} \to \mathbf{B}).$$

Note that the r.h.s may contain terms with negative signs since we have a 3d lattice and there are many nonintersecting path systems corresponding to non-identity permutations.

3.4. **Path transformations and sign-reversing involution.** In this subsection we describe certain path operations and a sign-reversing weight-preserving involution on $N(\mathbf{A}, \mathbf{B})$ which leaves only positive terms in (3). Suppose we have a nonintersecting path system $\mathbf{P} = (P_1, \ldots, P_n)$ from $\mathbf{A}$ to $\mathbf{B}$ where the paths are ordered with respect to the sinks ordering, i.e. P_i has sink B_i. We are going to introduce several important definitions necessary for path transformations.

Definition 6 (Projections and intersections)**.** Let us define the following notions:

(Path projections to planes) For a path P_i from A_j to B_i, the *projection* of P_i on the plane $z = k$ with $k \ge \mu_i'$ is defined as follows: take the first intersection point C of P_i with the plane $z = k$ and copy the part of P_i from the plane $z = \mu_i'$ to the plane $z = k$ starting from C and bounded by the plane $x = i$ so that when it hits that plane, it continues on it (by increasing the y-coordinate) to the point (i, m, k). See Fig. 2 (a). (If no such point C exists or if its x-coordinate is larger than i, then the projection is undefined. Note also that if the projection is defined, then $k \le \lambda_j' - 1$.)

(Projection intersections) We say that two paths *intersect on the plane $z = k$* if their projections on the plane $z = k$ are defined (as above) and intersect.

(Path system intersections) A path system has *no intersections on the plane $z = k$* if no pair of paths in this system intersects on the plane $z = k$.

Definition 7 (Cut edges)**.** When we project P_i on $z = k$, some (wall) part of P_i disappears on the projection. To keep track of such edges and (floor) edges of P_i lying between the planes $z = \mu_i'$ and $z = k$, we define *cut edges* lying on the plane $x = i$. Let $T_{i,k}$ be the first intersection point of the plane $x = i$ and the projection of P_i on the plane $z = k$ (whenever projection is defined). See again Fig. 2 (a). Note that for fixed i, the points $T_{i,k}$ have non-increasing y-coordinates. Let f_k be the (single floor) edge of P_i lying between the planes $z = k$ and $z = k+1$. We then define *k-th cut edge* of two types:
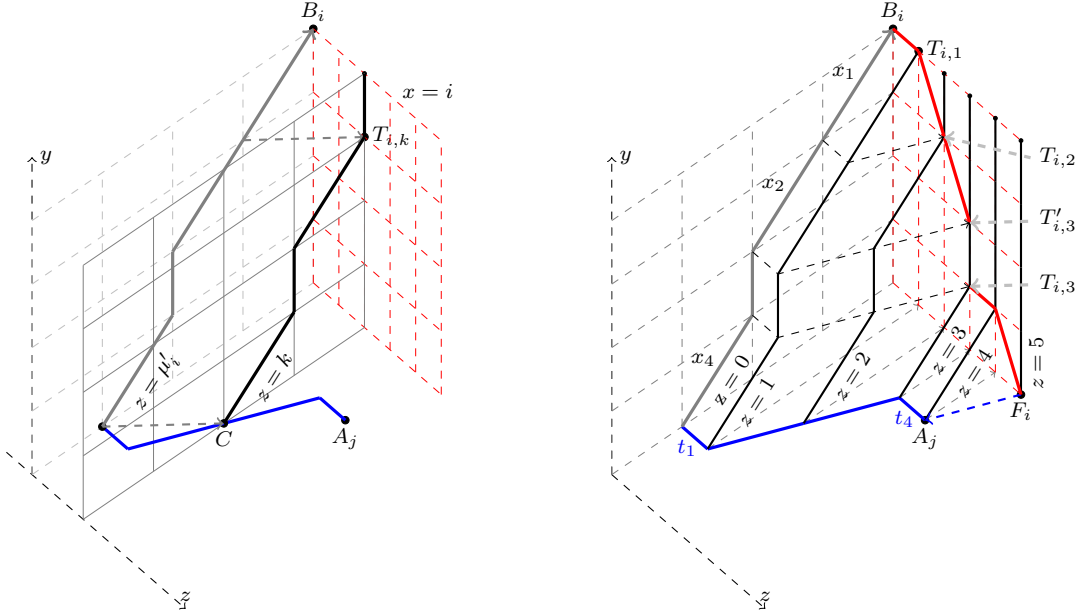
FIGURE 2. (a) Projection of a path $P_i : A_j \to B_i$ on the plane $z = k$. (b) Several projections (black solid) and cut edges (red solid).

(1) If $T_{i,k}$ and $T_{i,k+1}$ have the same y-coordinate, then we must have $w(f_k) = t_{k+1}$ and the segment $T_{i,k} T_{i,k+1}$ is the k-th cut edge of weight t_{k+1};

(2) Otherwise, $T_{i,k+1}$ is below $T_{i,k}$ (by y-coordinate), $w(f_k) = 1$, and there is a unique (wall) edge e_k of P_i of weight $w(e_k) = x_s$ for some s, such that this edge is seen in the projection of P_i on $z = k$, but disappears in the projection on $z = k+1$. Let $T'_{i,k+1} = T_{i,k} + (0, -1, 1)$. Then the segment $T_{i,k} T'_{i,k+1}$ is the k-th cut edge of weight x_s.

For k-th cut edges e and e' (corresponding to different i), e is *larger* than e' if the endpoint of e on the plane $z = k$ is higher than the endpoint of e' on the same plane.

Example 8. Consider Fig. 2 (b) where cut edges are shown as red solid segments. For $k = 0$ and the planes $z = 0$ and $z = 1$ we have $B_i = T_{i,0} = (4, 4, 0)$, $T_{i,1} = (4, 4, 1)$, $w(f_0) = t_1$. Then the 0-th cut edge is the segment $T_{i,0} T_{i,1}$ of weight t_1 (type 1). For $k = 2$ and the planes $z = 2$ and $z = 3$, we have $w(f_2) = 1$, $T_{i,2} = (4, 3, 2)$, $T'_{i,3} = (4, 2, 3)$, $T_{i,3} = (4, 1, 3)$ and $w(e_2) = x_2$. Then the 2-nd cut edge is the segment $T_{i,2} T'_{i,3}$ of weight x_2 (type 2).

Definition 9 (Step and slide operations). The operation step_k on $\mathbf{P}$ is defined as follows:

(1) choose minimal index i such that P_i and P_{i+1} intersect on the plane $z = k + 1$, but do not intersect on the plane $z = k$ (if there is no such index, do nothing);

(2) Let e_1 and e_2 be the edges of P_i and P_{i+1} between the planes $z = k$ and $z = k + 1$. By the choice of i, we must have the weights $w(e_1) = 1$ and $w(e_2) = t_{k+1}$ (other cases do not produce intersections on $z = k + 1$). Let C be the first common point of projections of P_i and P_{i+1} on the plane $z = k + 1$. Let e'_1 and e'_2 be the edges of projections of P_i and P_{i+1} on $z = k + 1$ preceding C (i.e. C is an endpoint of these edges). We must have
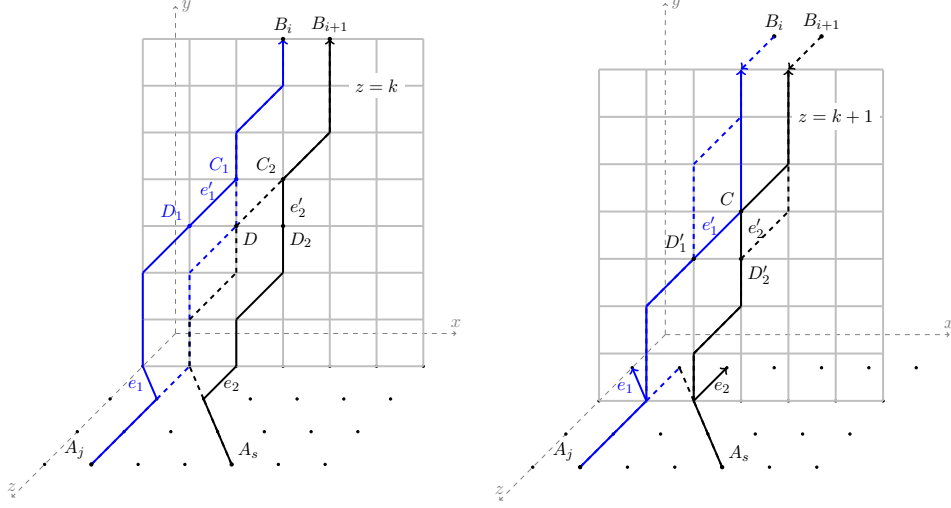
FIGURE 3. Paths P_i and P_{i+1} projected to the planes $z = k$ (left) and $z = k+1$ (right) before (solid) and after (dashed) applying the operation step_k.

$w(e_1') = x_s$ and $w(e_2') = 1$. We then modify these paths as follows. In P_i we switch the edge e_1 (of weight 1) to the edge of weight t_{k+1} with same starting point, shift the part of P_i between the edges e_1 and e_1' by 1 to the right (towards the x-axis), and switch the edge e_1' (of weight x_s) to the edge of weight 1 with the same ending point (other parts of P_i remain intact). The path P_{i+1} is modified in the same way; we 'exchange' edges e_1 with e_2, and 'exchange' e_1' with e_2'. See Fig. 3.

Define now the operation slide_k on $\mathbf{P}$ by repeatedly performing the operation step_k until it makes no changes in the path system. See Fig. 4.

Remark 1. One can observe that intersecting paths on each step_k are actually touching.

Definition 10 (Inverse steps and slide operations)**.** The operation rstep_k is defined as follows:
   (1) choose maximal index i such that P_i and P_{i+1} intersect on the plane $z = k$, but do not intersect on the plane $z = k + 1$, and k-th cut edge of P_i is not below k-th cut edge of P_{i+1}. (If there is no such index, do nothing.)
   (2) Let e_1 and e_2 be edges of P_i and P_{i+1} between the planes $z = k$ and $z = k + 1$. Again, by the choice of i we must have the weights $w(e_1) = t_{k+1}$ and $w(e_2) = 1$. Let D be the last common point of projections of P_i and P_{i+1} on $z = k$. Let e_1' and e_2' be the edges of projections of P_i and P_{i+1} to the plane $z = k$ succeeding D (i.e. D is an endpoint of these edges). We must have $w(e_1') = 1$ and $w(e_2') \neq 1$. We then 'exchange' e_1 with e_2, and 'exchange' e_1' with e_2', similarly as in the previous definition.

Define the operation slide_k^{-1} on path systems that do not intersect on the plane $z = k + 1$, by repeatedly performing the operation rstep_k until it makes no changes in the path system. See Fig. 4 in the reverse direction.

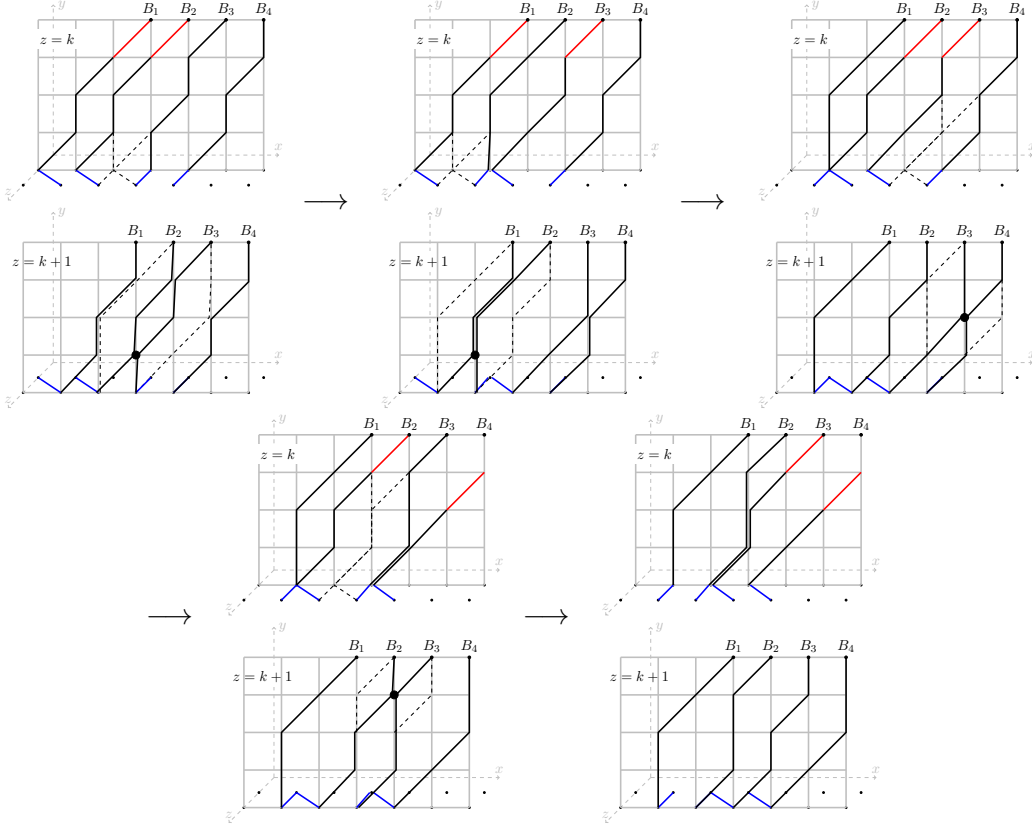**Lemma 11.** *The following properties hold:*

FIGURE 4. An example of slide_k operation where step_k is applied four times. Each step shows path system projections on the planes $z = k, k+1$ (top, bottom). The red edges disappear on $z = k + 1$. The dashed lines show how step_k is applied. When slide_k^{-1} is applied to the resulting system, it operates in the reverse direction and results in the initial path system.

(i) Suppose **P** has no intersections on the plane $z = k$. Then after performing slide_k, the paths with sinks on the planes $z = 0, \ldots, k$ have no intersections on the plane $z = k+1$.

(ii) Suppose **P** has no intersections on the plane $z = k + 1$ and k-th cut edges are non-increasing (from left to right). Then after performing slide_k^{-1}, the resulting system has no intersections on the plane $z = k$.

(iii) Suppose **P** has no intersections on the plane $z = k$. Then after performing slide_k and then slide_k^{-1} we get back to **P** (i.e. they are inverse operations).

Proof. It is illustrative to check the statements on Fig. 4. (i) First, let us show the that the following is true on each step_k: it takes two leftmost paths P_i and P_{i+1} which do not intersect on $z = k$ but intersect on $z = k + 1$, and turns them into paths that do not intersect on the plane $z = k + 1$ but intersect on $z = k$. Recall the intersection point C from Def. 9. Denote the projections of P_i and P_{i+1} on the plane $z = j$ ($j = k, k + 1$) as $P_i^{(j)}$ and $P_{i+1}^{(j)}$. Let C_1 and C_2 be the points of $P_i^{(k)}$ and $P_{i+1}^{(k)}$ that project to C on $z = k + 1$. Additionally, denote the

point D_1 as a preceding point of C_1 in $P_i^{(k)}$ and the point D_2 preceding C_2 in $P_{i+1}^{(k)}$, see Fig. 3. Observe that after the operation, the parts of $P_i^{(k)}$ and $P_{i+1}^{(k)}$ starting from the points C_1 and C_2 respectively, remain unchanged. Moreover, D_1 and D_2 will coincide, let us call this point by D, which will be the last common point of the paths $P_i^{(k)}$ and $P_{i+1}^{(k)}$ on $z = k$ after applying the operation. Now, let us show that $P_i^{(k+1)}$ and $P_{i+1}^{(k+1)}$ will not intersect after the operation step$_k$. Similarly, denote preceding points of C in $P_i^{(k+1)}$ and $P_{i+1}^{(k+1)}$ as D_1' and D_2', respectively. Then the part of the path $P_i^{(k+1)}$ until the point D_1' will stay unchanged; the same is true for D_2' and $P_{i+1}^{(k+1)}$. The parts starting from D_1' and D_2' will be nonintersecting as well, since they spread out in different directions.

So the application of step$_k$ makes two neighbouring paths (P_i, P_{i+1}) nonintersecting on $z = k + 1$ and intersecting on $z = k$. Assume that during the operation slide$_k$, we applied it to the pair of paths (P_i, P_{i+1}) for the first time, call this moment T. If the next pair to be processed is (P_{i-1}, P_i), then after step$_k(P_{i-1}, P_i)$ we will have the pair (P_i, P_{i+1}) nonintersecting on both planes $z = k, k + 1$. Furthermore, if one applies step$_k$ to the pair (P_{i+1}, P_{i+2}) for the first time after the moment T, then the pair (P_i, P_{i+1}) will not intersect on $z = k$ (but possibly on $z = k + 1$). Repeatedly combining this and the fact that the edges of weights t_{k+1} move to the left on each step$_k$, we see that the process slide$_k$ is finite and each pair of paths (P_i, P_{i+1}) will not intersect on $z = k + 1$ when slide$_k$ completes the action.

(ii) Since k-th cut edges are non-increasing from left to right, rstep$_k$ will have effect symmetric to step$_k$, one can observe that step$_k$ and rstep$_k$ are symmetric operations, if we reflect the lattice w.r.t. the line $x = 0$. Then in the above observations, the meaning of the points D and C will be swapped (i.e. D becomes the first point of intersection on the next plane); similarly, C_1 and C_2 will be swapped with D_1' and D_2', and so on. Therefore, slide$_k$ and slide$_k^{-1}$ are symmetric and the statement follows as in (i).

(iii) It is enough to note that slide$_k$ guarantees that after the operation, k-th cut edges will be non-increasing from left to right. Then by observing the symmetry in (ii) and the properties shown in (i), one can see that slide$_k^{-1}$ operates in the reverse direction as slide$_k$ and results in the initial path system. $\square$

Definition 12 (Transpose operation). Define the operation transpose$_k$ as follows:

(1) If there is no pair of paths intersecting on the plane $z = k$ such that one of them has a sink on this plane, then do nothing.

(2) Otherwise, let us consider intersection points between the paths with sinks on the plane $z = k$ and projections of other paths on $z = k$; among these intersections let C be the leftmost and lowest point. Let P_i and P_j be paths $(i < j)$ whose projections pass through C (so that P_i has a sink on $z = k$). Assume that P_j has a sink on the plane $z = k'$ for $k' \leq k$ and let $P_i : A_\ell \to B_i$, $P_j : A_r \to B_j$. Let also C' be the point of P_j that projects to C on the plane $z = k$ (preimage of C); E_i be the first common point of P_i with the plane $z = k$, and similarly define the point E_j for the path P_j; let F_j be the first common point of P_j and the plane $z = k'$. The defined points split the path P_i into
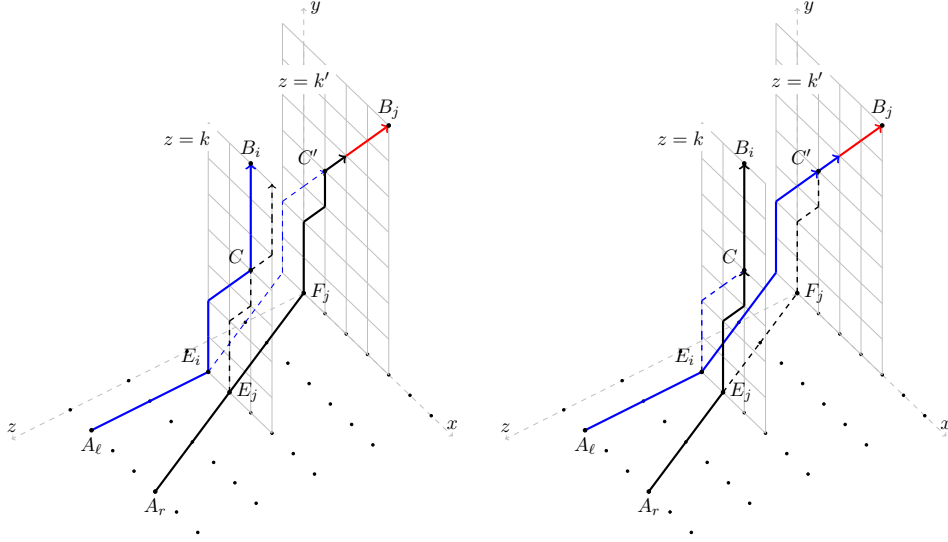
FIGURE 5. An example of transpose$_k$: before (left) and after (right). Note that wall edges (red) corresponding to cut edges between planes $z = k'$ and $z = k$ are unaffected.

three parts and P_j into four parts. Let us change P_i and P_j as follows:

$$P_i := P_i(A_\ell \to E_i)\, P_j(E_j \to F'_j)\, P_i(E_i \to C)\, P_j(C' \to B_j)$$
$$P_j := P_j(A_r \to E_j)\, P_j(F_j \to C')\, P_i(C \to B_i)$$

See Fig. 5.

Lemma 13. *The operation* transpose$_k$ *is an involution.*

Proof. Observe that before applying the transpose$_k$ and after applying it, the path projections on the plane $z = k$ remain the same. $\qquad\square$

We are now ready to state the definition of the main involution.

Definition 14 (The sign-reversing weight-preserving involution)**.** Denote

$$s_i := \text{slide}_i \cdot \text{slide}_{i-1} \cdot \ldots \cdot \text{slide}_0 \quad \text{and} \quad s_i^{-1} := \text{slide}_0^{-1} \cdot \text{slide}_1^{-1} \cdot \ldots \cdot \text{slide}_i^{-1}$$

The operation s_i expresses sequential sliding starting from the plane $z = 0$ all the way to the plane $z = i + 1$, where $i \in [0, \ell(\lambda) - 1]$. Define the map $\phi : N(\mathbf{A}, \mathbf{B}) \to N(\mathbf{A}, \mathbf{B})$ as follows. Let $\mathbf{P} \in N(\mathbf{A}, \mathbf{B})$. Choose minimal index $k \in [0, \ell(\lambda) - 1]$ such that $s_k^{-1} \cdot \text{transpose}_{k+1} \cdot s_k(\mathbf{P}) \neq \mathbf{P}$. If there is no such k, then $\phi(\mathbf{P}) := \mathbf{P}$. Otherwise, set

$$\phi(\mathbf{P}) := s_k^{-1} \cdot \text{transpose}_{k+1} \cdot s_k(\mathbf{P}).$$

Denote by I_ϕ the set of fixed points of ϕ, called *good* path systems.

Lemma 15. *We have: ϕ is a sign-reversing weight-preserving involution and*

(4) $$w(\mathbf{A} \to \mathbf{B}) = \sum_{\mathbf{P} \in I_\phi} w(\mathbf{P}).$$

Proof. Let $\mathbf{P} \in N(\mathbf{A}, \mathbf{B})$ and suppose $\mathbf{P}$ is not good. Observe that the map ϕ works as follows. First, we project the paths of $\mathbf{P}$ whose sinks are on the planes $z = 0, \ldots, k$, to the plane $z = k+1$ (by applying the operation s_k). By Lemma 11 (i), the resulting projections are nonintersecting on the plane $z = k + 1$. By the choice of k, the paths whose sinks are on the plane $z = k + 1$, have intersections with the projected paths (from the planes $z = 0, \ldots, k$); we then perform the transposition operation on certain pair of paths intersecting on $z = k + 1$ (by applying the operation transpose$_{k+1}$). The operation s_k guarantees non-increasing order of cut edges (from left to right), and transpose$_{k+1}$ preserves this property. Then we perform backward slides preserving the absence of path intersections (by applying the operation s_k^{-1}). By Lemma 11 (ii), the backward slides guarantee that the paths will not intersect. Now, by Lemma 11 (iii) and Lemma 13 it is then clear that ϕ is a sign-reversing weight-preserving involution. Notice also that good path systems correspond to the identity permutation and hence have positive sign (otherwise the paths creating an inversion must have intersection on at least one of the planes containing the sinks of these paths). Hence (4) follows. $\square$

3.5. **Final step: the correspondence between good path systems and skew RPP.** Let us now take any good path system $\mathbf{P}$. We are going to describe how to obtain a unique RPP with the same $(\mathbf{t}, \mathbf{x})$ weight.

Denote by $\mathbf{P}'$ the path system obtained from $\mathbf{P}$ by (slightly) changing the sources to the points $F_i = (i, 0, \lambda_i')$ and adding single steps $F_i \to A_i = (i - 1, 0, \lambda_i' - 1)$. Let us consequently perform the operations slide$_0, \ldots,$ slide$_{\ell(\lambda)-1}$ on it. On each operation slide$_k$ during this process, let us keep track of the k-th cut edges belonging to the planes $x = i$ for $i = \mu_{k+1} + 1, \ldots, \lambda_{k+1}$, when we project paths from $z = k$ to $z = k + 1$. Note that after the operation slide$_k$, k-th cut edges do not change in further slide operations. Then for i fixed, k-th cut edges (for $k \in [\mu_i', \lambda_i' - 1]$) belong to the path R_i on the plane $x = i$ from the point B_i down to F_i, formed by following cut edges and (vertical) edges $T_{i,k} T_{i,k}'$ (where type 1 cut edges have t-weights, type 2 cut edges have x-weights, and vertical edges have weights 1, see Fig. 2 (b)).

Given the path R_i, let us record the i-th column (for $i = 1, \ldots, n$) of a skew RPP of shape λ/μ as follows: from top to bottom, we add an entry s if we see an edge of weight x_s (type 2 cute edge) and add an empty entry if we see an edge of t weight (type 1 cut edge). The resulting filling might have some empty entries. To form a proper RPP we then fill each empty entry by the first nonempty entry below it in the same column. The resulting tableau is an RPP with the weight $\prod_i x_i^{c_i} \prod_j t_j^{d_j}$ which is also the weight of $\mathbf{P}$. Since each operation slide$_k$ preserves the non-increasing order of cut edges at this step, we have R_i is not below R_{i+1} (by y-coordinate), which explains why it is a proper RPP. For an example of this procedure, see Fig. 6.

Conversely, given any RPP of shape λ/μ we are now describing an inverse procedure how to reconstruct $\mathbf{P}$. From given RPP it is easy restore the paths $R_i : B_i \to F_i$ by reading its i-th column. Note that in our lattice there is a unique path P_i' of weight $w(R_i)$ from F_i to B_i. Since R_i is never lower than R_{i+1} for all i, k-th cut edges will not increase (from left to right) for each k. Combining this with the fact that P_i' do not intersect on the initial plane $z = \ell(\lambda)$ (n vertical lines), one deduces with Lemma 11 that consequent application of the inverse operations slide$_{\ell(\lambda)-1}^{-1}, \ldots,$ slide$_0^{-1}$ to $\mathbf{P}'$ will result in a good path system $\mathbf{P}$ of the same weight. This completes the proof.
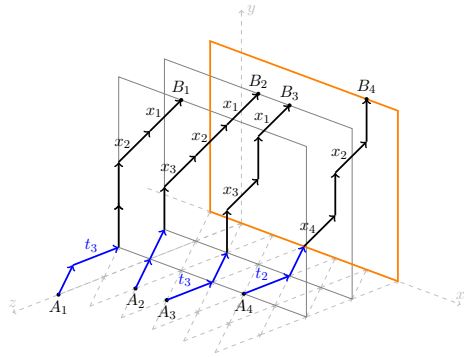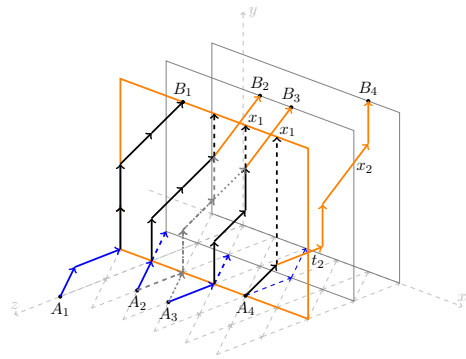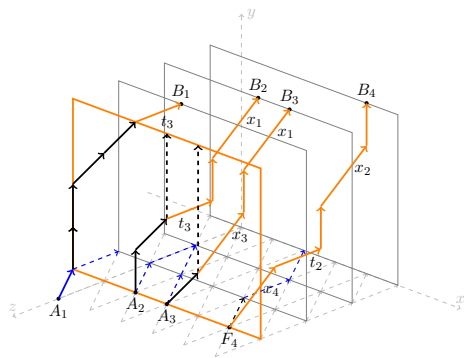
(A) Initial good path system **P**
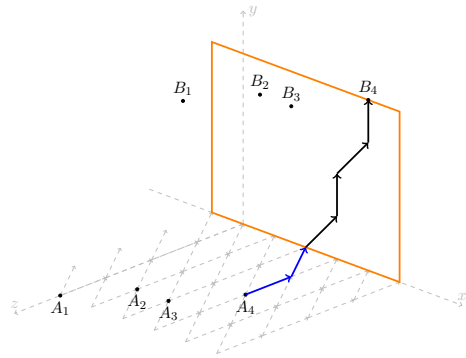
(B) Before slide$_0$, only P_4 is affected.
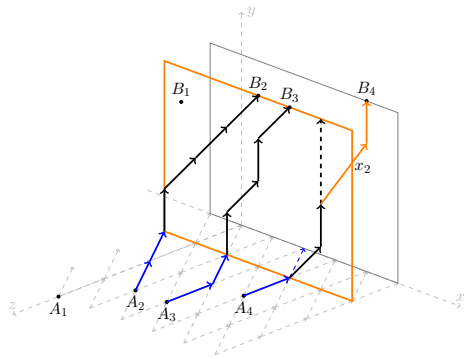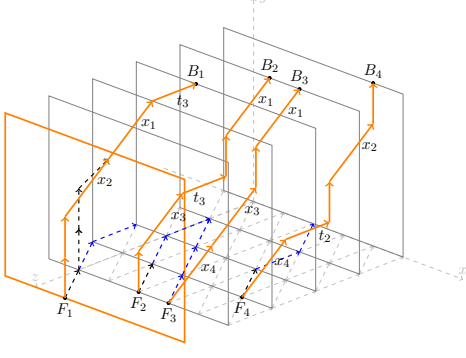
(C) slide$_0$ applied, project on $z = 1$. Cut edges (orange) of each slide are shown on $x = i$.

(D) slide$_1$ applied, project on $z = 2$. Gray path results from step$_2(P_2, P_3)$.

(E) slide$_2$ applied, project on $z = 3$.

(F) slide$_3$ and slide$_4$ applied. Each orange path (on some $x = i$) converts to a column of RPP.

(G) A reduced RPP: empty boxes of t weights.

(H) Resulting RPP

FIGURE 6. From a good path system to RPP.

Remark 2. The sliding operations on paths are in fact related to classical *jeu de taquin* type operations which can be applied for RPP. This is also related to Schur expansions of dual Grothendieck polynomials.

Remark 3. While certain components of our proof look somewhat complicated and technical, they are quite natural, and we expect that they may have more applications. A non-planar 3d construction is unusual in these type of problems and it would be interesting to see if it has other consequences.

Remark 4. Our proof is new even in the straight case shape $\mu = \varnothing$. The proof in [Yel17] relied on planar construction combined with the Schur expansion of g_λ from [LP07] (based on RSK). Here, in this special case, the sinks are all on the plane $z = 0$, all nonintersecting path systems are good (so there is no need to apply the involution), and we then apply sequential slide transformations to get RPP's.

4. The dual formula

Set $h_0 = 1$ and $h_i = 0$ for $i < 0$. Let Λ be the ring of symmetric functions. Define the automorphism $\varphi : \Lambda \to \Lambda$ via the complete homogeneous symmetric functions $\{h_n\}$ as follows:

$$\varphi(h_n) = h_n + h_{n-1} + \cdots + h_0, \quad n \geq 1.$$

It is easy to check that we have

$$\varphi^\ell h_n = \sum_{k=0}^n \binom{\ell + k - 1}{k} h_{n-k}, \quad \ell, n \in \mathbb{Z}.$$

Note also that $\varphi^{-1} h_n = h_n - h_{n-1}$ and for $\ell \geq 0$ we have $\varphi^\ell h_n(\mathbf{x}) = h_n(1^\ell, \mathbf{x})$.

Theorem 16. *Let $n \geq \ell(\lambda), \ell(\mu)$. The following determinantal identity holds*

(5) $$g_{\lambda/\mu} = \det \left[\varphi^{i-j} h_{\lambda_i - i - \mu_j + j} \right]_{1 \leq i,j \leq n}.$$

Proof. It is known that that $\{g_\lambda\}$ is a basis of Λ and there is an involutive automorphism (see [Buc02, Cor. 6.7], [Yel19, Prop. 7.3]) $\tau : \Lambda \to \Lambda$ such that for all λ, μ

$$\tau(g_{\lambda/\mu}) = g_{\lambda'/\mu'}$$

This involution can be defined via the generators of Λ as follows

$$\tau : g_{(k)} = h_k \longmapsto g_{(1^k)} = e_k(1^{k-1}, \mathbf{x}) = \sum_\ell \binom{k-1}{k-\ell} e_\ell.$$

Let us now expand the entries of the determinant (5)

$$\varphi^{i-j} h_{\lambda_i - i - \mu_j + j} = \sum_k \binom{\lambda_i - \mu_j - k - 1}{\lambda_i - i - \mu_j + j - k} h_k.$$

After applying the involution τ we get the following expression

$$\tau(\varphi^{i-j}h_{\lambda_i-i-\mu_j+j}) = \sum_k \binom{\lambda_i-\mu_j-k-1}{\lambda_i-i-\mu_j+j-k}\sum_\ell \binom{k-1}{k-\ell}e_\ell$$
$$= \sum_\ell e_\ell \sum_k \binom{\lambda_i-\mu_j-k-1}{\lambda_i-i-\mu_j+j-k}\binom{k-1}{k-\ell}.$$

By comparing the coefficients at $[t^{\lambda_i-i-\mu_j+j-\ell}]$ from both sides of the identity $(1-t)^{j-i-\ell} = (1-t)^{j-i}(1-t)^{-\ell}$ it is easy to see that the following identity holds

$$\sum_k \binom{\lambda_i-\mu_j-k-1}{\lambda_i-i-\mu_j+j-k}\binom{k-1}{k-\ell} = \binom{\lambda_i-1-\mu_j}{\lambda_i-i-\mu_j+j-\ell}.$$

Hence

$$\tau(\varphi^{i-j}h_{\lambda_i-i-\mu_j+j}) = \sum_\ell \binom{\lambda_i-1-\mu_j}{\lambda_i-i-\mu_j+j-\ell}e_\ell = e_{\lambda_i-i-\mu_j+j}(1^{\lambda_i-1-\mu_j},\mathbf{x}).$$

and therefore using Theorem 1 we have

$$\tau\det\left[\varphi^{i-j}h_{\lambda_i-i-\mu_j+j}\right]_{1\le i,j\le n} = \det\left[\tau(\varphi^{i-j}h_{\lambda_i-i-\mu_j+j})\right]_{1\le i,j\le n} = g_{\lambda'/\mu'}$$

which establishes the needed identity. $\square$

Remark 5. Note that entries of the matrix in this dual formula may contain linear combinations of $\{h_n\}$ with negative terms. It would be interesting to find a positive dual formula for $g_{\lambda/\mu}$. It would also be interesting to find a combinatorial proof of this dual formula.

4.1. **Straight shape case.** We obviously we have the following special case for $\mu = \varnothing$.

Corollary 17. *We have*

(6) $$g_\lambda = \det\left[\varphi^{i-j}h_{\lambda_i-i+j}\right]_{1\le i,j\le \ell(\lambda)}.$$

There is also the following *positive* formula that holds for g_λ (which is not that obvious from the above identity).

Corollary 18. *We have*

(7) $$g_\lambda(\mathbf{x}) = \det\left[\varphi^{i-1}h_{\lambda_i-i+j}\right]_{1\le i,j\le \ell(\lambda)} = \det\left[h_{\lambda_i-i+j}(1^{i-1},\mathbf{x})\right]_{1\le i,j\le \ell(\lambda)}.$$

This formula follows from (6) by elementary (column) transformations; we omit these details. Alternatively, it can be proved via a lattice-path construction similar to the one used in [Yel17] (which is planar and simpler than our main construction here). This formula will be used for deriving one more determinantal identity given in the next section.

5. Bialternant formula

Theorem 19. *Let $n \ge \ell(\lambda)$. The following formula holds*

$$g_\lambda(x_1,\ldots,x_n) = \frac{\det\left[h_{\lambda_i+n-i}(1^{i-1},x_j)\right]_{1\le i,j\le n}}{\prod_{i<j}(x_i-x_j)}$$

where

$$h_m(1^{i-1}, x) = \sum_{k=0}^{m} \binom{i+k-2}{k} x^{m-k}.$$

Proof. The proof follows along the same lines as the classical derivation that relates Jacobi-Trudi and bialternant formulas for Schur polynomials, see [Mac98, (3.6)].

Let $\mathbf{x} = (x_1, \ldots, x_n)$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ and consider the matrices

$$H_\alpha := \left[h_{\alpha_i - n + k}(1^{i-1}, \mathbf{x}) \right]_{1 \le i, k \le n}, \qquad E := \left[(-1)^{n-k} e_{n-k}^{(j)} \right]_{1 \le k, j \le n},$$

where the second matrix is defined so that

$$E_j(t) := \sum_{\ell=0}^{n-1} e_\ell^{(j)} t^\ell := \prod_{i \ne j} (1 + x_i t).$$

Let

$$H_i(t) := \sum_{k=0}^{\infty} h_k(1^{i-1}, \mathbf{x}) \, t^k = (1-t)^{1-i} \prod_k (1 - x_k t)^{-1}.$$

Thus

$$H_i(t) \cdot E_j(-t) = (1-t)^{1-i} (1 - x_j t)^{-1} = \sum_m h_m(1^{i-1}, x_j) t^m$$

and by comparing the coefficients at $[t^{\alpha_i}]$ from both sides of this equality we obtain that

$$\sum_{k=1}^{n} h_{\alpha_i - n + k}(1^{i-1}, \mathbf{x})(-1)^{n-k} e_{n-k}^{(j)} = h_{\alpha_i}(1^{i-1}, x_j).$$

Therefore,

$$H_\alpha \cdot E = [h_{\alpha_i}(1^{i-1}, x_j)]_{1 \le i, j \le n}, \qquad \det[H_\alpha] \cdot \det[E] = \det[h_{\alpha_i}(1^{i-1}, x_j)]_{1 \le i, j \le n}$$

Note that $\det[E] = \prod_{i < j}(x_i - x_j)$ and the sequence $\alpha = (\alpha_1, \ldots, \alpha_n)$ given by $\alpha_i = \lambda_i + n - i$ satisfies $\det[H_\alpha] = g_\lambda$ by the identity (7), and hence we get the needed formula. $\square$

Remark 6. Even though we state the formula for $n \ge \ell(\lambda)$, the definition of $g_\lambda(x_1, \ldots, x_n)$ allows to compute the polynomial for $n < \ell(\lambda)$ as well. For example, $g_\lambda(x_1) = x_1^{\lambda_1}$.

6. Some open questions

6.1. **Ribbon formulas.** Schur function determinants have a beautiful unifying theory via ribbon decompositions known as the Hamel-Goulden formula [HG95], see also [CYY05]. It unifies Jacobi-Trudi types, Giambelli hook (see e.g. [Mac98]), and Lascoux-Pragacz [LP88] ribbon formulas. Is there analogous formula for dual Grothendieck polynomials that would generate such identities? Note that there is a nontrivial Giambelli-type identity for g_λ obtained in [LN14].

6.2. **Positive specializations.** An algebra homomorphism (specialization) $\rho : \Lambda \to \mathbb{R}$ is called *g-positive* if $\rho(g_{\lambda/\mu}) \ge 0$ for all λ, μ. The characterization problem for g-positive specializations studied in [Yel20] can now be restated via the positivity of determinants:

$$\det \left[\rho(e_{\lambda_i - i - \mu_j + j}(1^{\lambda_i - \mu_j - 1}, \mathbf{x})) \right] \ge 0.$$

This form differs from totally nonnegative Toeplitz matrices in the Schur case.

6.3. **Canonical deformations.** Dual Grothendieck polynomials have two-parameter deformations $g_{\lambda/\mu}^{(\alpha,\beta)}$ introduced in [Yel17], which satisfy the following properties: $g_{\lambda/\mu}^{(0,1)} = g_{\lambda/\mu}$ and the self-duality $\omega(g_{\lambda/\mu}^{(\alpha,\beta)}) = g_{\lambda'/\mu'}^{(\beta,\alpha)}$ under the standard involution $\omega : h_n \mapsto e_n$ of the ring Λ. It would be interesting to find skew Jacobi-Trudi-type identities for these deformations as well.

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